2.12 - Monads on **Set**

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired.

In Section 10, we classified algebraic categories, and showed that each one is isomorphic to some variety in universal algebra. In this section we will do something surprisingly easier: we will take an arbitrary monad (T, η, μ) on the category **Set**, and show that it is the monad sending $X \to F_S(\Omega, X)$ for some variety $\mathcal{V}(S)$. This, in effect, associates any adjunction (G, F, η) with $F : \mathbf{C} \to \mathbf{Set}$ with a monadic adjunction involving a variety.

There is one obstacle of this, though: the variety might have infinitary operations if we are not careful. For example, the least upper bound in a complete join-semilattice is infinitary — it allows infinitely many operands at once. We have always assumed varieties consist of *finitary* operations, so we must first define a condition on the monad.

DEFINITION *A monad (T, η, μ) on* **Set** *is said to be **finitary** provided that whenever X is a set and $w \in TX$, there exists a finite subset X' of X such that if $\iota : X' \to X$ is the inclusion map, then $T(\iota) : TX' \to TX$ has w in its image.*

This statement says that any expression in TX uses only finitely many symbols. Thus it guarantees the "finiteness" of expressions and operators, and is related to Lemma 1.20 in the previous chapter. We now state and prove our theorem.

THEOREM 2.11 *If (T, η, μ) is a finitary monad on* **Set***, then there exists a variety $\mathcal{V}(S)$ such that (T, η, μ) is the monad sending $X \to F_S(\Omega, X)$, in Example 1 of Section 11.*

(The theorem also holds if the monad is not finitary, but then there would be infinitary operators in $\mathcal{V}(S)$, which is beyond our course.)

Proof of Theorem 2.11. First we define a universal-algebra signature Ω. For each $n \geq 0$, set $\Omega(n) = T\{x_1, x_2, \ldots, x_n\}$. In particular, $\Omega(0) = T\emptyset$.

Now, for each set X we form the following Ω-algebra structure on TX: For $\omega \in \Omega(n)$, $a_1, a_2, \ldots, a_n \in TX$, let φ be the map $x_i \to a_i$ from $\{x_1, x_2, \ldots, x_n\} \to TX$. Then $\mu_X T(\varphi)$ goes from $\Omega(n) \to TX$; define $(\omega a_1 a_2 \ldots a_n) = \mu_X T(\varphi)(\omega)$. In particular, if $n = 0$, let φ be the unique map $\emptyset \to TX$ and define $(\omega_{TX}) = \mu_X T(\varphi)(\omega)$. We thus have made each TX into an Ω-algebra. It is evident that if $\omega \in \Omega(n)$ then $(\omega \eta_X(x_1) \eta_X(x_2) \ldots \eta_X(x_n))$ is the element ω of $T\{x_1, x_2, \ldots, x_n\}$.

We now form a set $S \subseteq F(\Omega, X_0)^2$ of identities for a variety $\mathcal{V}(S)$. [Recall that X_0 is a countably infinite set.] For any expressions $w_1, w_2 \in F(\Omega, X_0)$, we have $w_1, w_2 \in F(\Omega, X')$ for some finite subset of $X' = \{x_1, x_2, \ldots, x_n\}$ of X_0. Thus w_1 and w_2 are expressions in $x_1, x_2, \ldots, x_n$, and therefore, when each x_i

is changed to $\eta_{X'}(x_i)$ the resulting expressions can be evaluated to elements of the algebra $TX' = \Omega(n)$; if they are the same element, assign $(w_1, w_2) \in S$.

We now claim that $\mathcal{V}(S)$ is our desired variety; to do this, we must show four things:

(1) For any set X, TX is identifiably $F_S(\Omega, X)$;
(2) For any $f : X \to Y$; $T(f) : TX \to TY$ is the homomorphism $f : F_S(\Omega, X) \to F_S(\Omega, Y)$ which extends f;
(3) $\eta_X : X \to TX$ is the usual inclusion $X \to F_S(\Omega, X)$;
(4) $\mu_X : TTX \to TX$ is the evaluation homomorphism $F_S(\Omega, TX) \to TX$.

This will mean the monad necessarily matches up with the one we have in the previous section.

We first show that for each $f : X \to Y$, $T(f) : TX \to TY$ is a homomorphism. Take any $\omega \in \Omega(n), a_1, a_2, \ldots, a_n \in TX$. Let $g_1 : \{x_1, x_2, \ldots, x_n\} \to TX$ sending $x_i \to a_i$, so $g_2 = T(f)g_1 : \{x_1, x_2, \ldots, x_n\} \to TY$ sends $x_i \to T(f)(a_i)$. Then by definition of ω, $\mu_X T(g_1)(\omega) = (\omega a_1 a_2 \ldots a_n)$, and $\mu_X T(g_2)(\omega) = (\omega T(f)(a_1)T(f)(a_2)\ldots T(f)(a_n))$. However, $\mu_X T(g_2) = \mu_X T(T(f)g_1) = \mu_X TT(f)T(g_1) = T(f)\mu_X T(g_1)$, by naturality of μ. Thus $\mu_X T(g_2)(\omega) = T(f)[\mu_X T(g_1)(\omega)] = T(f)(\omega a_1 a_2 \ldots a_n)$, and

$$T(f)(\omega a_1 a_2 \ldots a_n) = (\omega T(f)(a_1)T(f)(a_2)\ldots T(f)(a_n)),$$

proving that $T(f)$ is a homomorphism.

Next, we prove that $\mu_X : TTX \to TX$ is a homomorphism. For any $\omega \in \Omega(n)$, $a_1, a_2, \ldots, a_n \in TTX$, let $g_1 : \{x_1, x_2, \ldots, x_n\} \to TTX$ send $x_i \to a_i$ and $g_2 = \mu_X g_1 : \{x_1, x_2, \ldots, x_n\} \to TX$. Then by definition of ω, $\mu_{TX} T(g_1)(\omega) = (\omega a_1 a_2 \ldots a_n)$, and $\mu_X T(g_2)(\omega) = (\omega \mu_X(a_1)\mu_X(a_2)\ldots\mu_X(a_n))$. But $\mu_X T(g_2) = \mu_X T(\mu_X g_1) = \mu_X T(\mu_X)T(g_1) = \mu_X \mu_{TX} T(g_1)$ [recall that $\mu(T\mu) = \mu(\mu T)$]. Therefore, $\mu_X T(g_2)(\omega) = \mu_X[\mu_{TX} T(g_1)(\omega)] = \mu_X(\omega a_1 a_2 \ldots a_n)$. It follows that μ_X is a homomorphism, as $\mu_X T(g_2)(\omega)$ is equal to both $\mu_X(\omega a_1 a_2 \ldots a_n)$ and $(\omega \mu_X(a_1)\mu_X(a_2)\ldots\mu_X(a_n))$.

We are now ready to prove statements (1)-(4) above.

To show (1), first we need to prove that the Ω-algebra TX is in $\mathcal{V}(S)$. To do this, take any homomorphism $\varphi : F(\Omega, X_0) \to TX$ and $(w_1, w_2) \in S$. Then $w_1, w_2 \in F(\Omega, X')$ for some finite subset $X' = \{x_1, x_2, \ldots, x_n\}$ of X_0. Furthermore, $w_1, w_2 \in \Omega(n)$, and if $a_i = \varphi(x_i) \in TX$ for each i, then $\varphi(w_j) = (w_j a_1 a_2 \ldots a_n)$ for $j = 1, 2$ because φ is a homomorphism. Now take the map $\psi : X' \to TX$ sending $x_i \to a_i$ and form $\phi = \mu_X T(\psi) : TX' \to TX$. The last two paragraphs imply that ϕ is a homomorphism, so $\phi(w_j x_1 x_2 \ldots x_n) = (w_j a_1 a_2 \ldots a_n)$ for $j = 1, 2$. However, $(w_1 x_1 x_2 \ldots x_n)$ and $(w_2 x_1 x_2 \ldots x_n)$ are the same element of TX', because $(w_1, w_2) \in S$. Therefore, $(w_1 a_1 a_2 \ldots a_n) = (w_2 a_1 a_2 \ldots a_n)$, from which it follows that $(w_1, w_2) \in F(\Omega, X_0)^2$ is in the kernel of φ. Therefore, $TX \in \mathcal{V}(S)$.

Because of this, there is a unique homomorphism $h_X : F_S(\Omega, X) \to TX$ extending the set map $\eta_X : X \to TX$. We claim that h_X is an isomorphism. To support this claim, first suppose X is a finite set, say $\{x_1, x_2, \ldots, x_n\}$. Then if $h_X(\overline{e}_1) = h_X(\overline{e}_2)$ with $e_1, e_2 \in F(\Omega, X)$, then after identifying X with a subset of X_0, e_1 and e_2 are expressions in $x_1, x_2, \ldots, x_n$ which evaluate, after substituting $x_i \to \eta_X(x_i)$, to the same element in $TX = \Omega(n)$. Hence $(e_1, e_2) \in S$ by definition of S and $\overline{e}_1 = \overline{e}_2$. Therefore, h_X is injective. For each $\omega \in TX = \Omega(n)$, h_X sends the expression $(\omega x_1 x_2 \ldots x_n)$ to $(\omega \eta_X(x_1)\eta_X(x_2)\ldots\eta_X(x_n)) = \omega$ (because h_X is a homomorphism), so h_X is surjective. Thus h_X is an isomorphism if X is finite.

Now suppose X is infinite. For each finite subset X' of X, recall that $F_S(\Omega, X')$ is a subalgebra of $F_S(\Omega, X)$, and realize that TX' is identifiably a subalgebra of TX, because if $\iota : X' \to X$ is the inclusion map, $T(\iota)$ is injective, and is a homomorphism as proved above. It is clear that h_X sends elements of $F_S(\Omega, X')$ to elements of TX'. Hence $h_X|F_S(\Omega, X')$ is the homomorphism from $F_S(\Omega, X') \to TX'$ sending $x \in X'$ to $\eta_X(x) = \eta_{X'}(x)$; by the argument in the previous paragraph, it is an isomorphism. Thus whenever $h_X(\overline{e}_1) = h_X(\overline{e}_2)$, the restriction of h_X to $F_S(\Omega, X')$ with some suitable finite subset X' sends $\overline{e}_1$ and $\overline{e}_2$ to the same element of TX', so that $\overline{e}_1 = \overline{e}_2$ and h_X is injective. The surjectivity of h follows from the fact that (T, η, μ) is finitary — for all $\omega \in TX$, $\omega \in TX'$ for some finite subset X' of X, and therefore h sends an element of $F_S(\Omega, X')$ to ω.

This proves that h_X is an isomorphism, and it identifies TX with $F_S(\Omega, X)$. The proof of (1) is complete.

(3) follows from the fact that if $i_X : X \to F_S(\Omega, X)$ is the usual free-algebra inclusion, then $h_X i_X = \eta_X$. Thus when h_X identifies the algebras together, it identifies i_X and η_X together.

As for statement (2), we already know that $T(f) : TX \to TY$ is a homomorphism, so we need only show that $T(f)$ sends $\eta_X(x), x \in X$ to $\eta_Y(f(x))$. This is an immediate consequence of the naturality of η, which implies $T(f)\eta_X = \eta_Y f$.

Now to prove statement (4): we already proved that μ_X is a homomorphism, so we only need to show that it extends the identity map $TX \to TX$; that is, $\mu_X \eta_{TX} = 1_{TX}$. But this follows immediately from the monad axiom $\mu(\eta T) = 1_T$.

Thus statements (1)-(4) are proven and the proof is completed. ■

It's not easy to overestimate the power of what we have just proved. We've shown that from (almost) every monad on **Set**, we can recover a variety in universal algebra. There may be many other adjunctions from which monads on **Set** are formed; but those adjunctions would most likely not be monadic.

EXERCISES

1. Prove or disprove:

   (a) If (T, η, μ) is a monad on **Set**, then its Eilenberg-Moore category $\mathbf{C}^T$ is an algebraic category [Section 10].

   (b) (T, η, μ) is a finitary monad if and only if $\mathbf{C}^T$ is a finitary algebraic category.

2. Tell whether or not each monad on **Set** is finitary. If it is, state which variety it comes from. If not, find some other way to describe the Eilenberg-Moore category.

   (a) $TX = \mathcal{P}(X)$; for $f : X \to Y$, $T(f)$ is the image map $\mathcal{P}(X) \to \mathcal{P}(Y)$ sending $S \to f(S)$; $\eta_X : X \to \mathcal{P}(X)$ sends each $x \in X$ to $\{x\}$; $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ sends each set of subsets of X to the union of the subsets. [Example 3 of Section 11]

   (b) Same as part (a), but TX is the set of nonempty subsets of X.

   (c) Same as part (a), but TX is the set of finite subsets of X.

   (d) Same as part (a), but TX is the set of countable subsets of X.

   (e) Let M be a fixed monoid. $TX = M \times X$; for $f : X \to Y$, $T(f)$ is the map $M \times X \to M \times Y$ sending $(m, x) \to (m, f(x))$; $\eta_X : X \to M \times X$ sends each $x \in X$ to $(1, x)$; $\mu_X : M \times (M \times X) \to M \times X$ sends $(m_1, (m_2, x)) \to (m_1 m_2, x)$. [Example 2 of Section 11]

   (f) $TX = \{\circ\}$ for all X and $T(f), \eta, \mu$ are defined in the unique ways.

   (g) $TX = X \times X$; for $f : X \to Y$, $T(f)$ is the map $X \times X \to Y \times Y$ sending $(x_1, x_2) \to (f(x_1), f(x_2))$; $\eta_X : X \to X \times X$ sends each $x \in X$ to (x, x); $\mu_X : (X \times X) \times (X \times X) \to X \times X$ sends $((x_1, x_2), (x_3, x_4))$ to (x_1, x_4).

   (h) Let S be a fixed set. $TX = X^S$ [functions from S to X]; for $f : X \to Y$, $T(f)$ is the map $X^S \to Y^S$ sending $h \to fh$ for $h : S \to X$; $\eta_X : X \to X^S$ sends each $x \in X$ to the constant function $s \to x$ in X^S; $\mu_X : (X^S)^S \to X^S$ sends each $h : S \to X^S$ to the map $s \to h(s)(s)$ from $S \to X$. [*Caution*: Whether this monad is finitary depends on something about S. Parts (f) and (g) are special cases of this, so they may help.]

   (i) Let S be a fixed set. $TX = X \uplus S$; for $f : X \to Y$, $T(f)$ is the map $X \uplus S \to Y \uplus S$ sending each $x \in X$ to $f(x)$ and each element of S to itself; $\eta_X : X \to X \uplus S$ sends each x to itself in the disjoint union summand X; $\mu_X : (X \uplus S) \uplus S = X \uplus S$ sends each x to itself and each s in either of the S's to itself in the summand S.