2.11 - Monads

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired.

Recall what expressions are in universal algebra. They comprise the free algebras, and are the most "abstract" way to apply operations to arbitrary symbols. The three fundamental things about an expression are that:

(1) Each symbol is in some way a primitive expression.

(2) An expression of expressions can be "flattened" into a great expression.

(3) Symbol substitutions can be made anywhere in the expressions.

And there are two additional facts of coherence:

(4) Given an expression of expressions of expressions [triple-layer!], it doesn't make a difference whether you flatten them starting from the inner layers or the outer layers.

(5) A primitive expression consisting of one expression flattens into the expression, and flattening an expression of primitive expressions yields the same expression with the symbols.

To see what those statements mean, let $\mathcal{V}(S)$ be a variety, then define $T : \mathbf{Set} \to \mathbf{Set}$ as follows. For each set X, define TX to be the set $F_S(\Omega, X)$. Then every set map $f : X \to Y$ yields a unique homomorphism $\varphi : F_S(\Omega, X) \to F_S(\Omega, Y)$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ F_S(\Omega, X) \arrow[r, "\varphi"] & F_S(\Omega, Y) \end{tikzcd}$$

is commutative, so define $T(f)$ to be φ regarded as a set map. It is clear then that T is a functor. $T(f)$ does the job of "substituting symbols" in each expression in X, by where the map f sends them to Y.

Now for each X, let $\eta_X = i_X : X \to TX$. The above diagram and how T is defined immediately implies that η is a natural transformation $1_{\mathbf{Set}} \Rightarrow T$. η is the device which "sees each symbol as a primite expression," sending each element of X to the length-one expression in $F_S(\Omega, X)$.

Next, notice that if A is a $\mathcal{V}(S)$-algebra, the identity map $1_A : A \to A$, where the first A is regarded as a set and the second as an algebra, extends to a unique homomorphism $e_A : F_S(\Omega, A) \to A$. It is called the **evaluation map** of A and assigns every expression in A to the value A gives it. It is clear that for every homomorphism $f : A \to B$,

$$\begin{tikzcd} F_S(\Omega, A) \arrow[r, "T(f)"] \arrow[d, "e_A"'] & F_S(\Omega, B) \arrow[d, "e_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is commutative due to f being a homomorphism.

Since the above holds for *any* algebras A and B and homomorphism f, it holds if $A = F_S(\Omega, X), B = F_S(\Omega, Y)$ and f is the homomorphism φ mentioned above. It follows that if each μ_X is assigned to be $e_{F_S(\Omega,X)} : F_S(\Omega, F_S(\Omega, X)) \to F_S(\Omega, X)$, μ is a natural transformation $TT \Rightarrow T$. μ is the device which takes each expression of expressions, and evaluates the expression in the free algebra, thus "flattening" the expression into one great expression.

Thus we have a triple (T, η, μ) where η devices fact (1) about an expression, μ devices fact (2), and the functorial nature of T devices fact (3). [Here's a nice little exercise: informally, what does the naturality of η and μ say about the expressions?]

Now, note that for any algebra A, the fact that e_A is a *homomorphism* implies that

$$\begin{tikzcd} TTA \arrow[r, "T(e_A)"] \arrow[d, "e_{TA}"'] & TA \arrow[d, "e_A"] \\ TA \arrow[r, "e_A"] & A \end{tikzcd}$$

is commutative. Taking TX for A and noting that $\mu_X = e_{TX}$, the diagram becomes

$$\begin{tikzcd} TTTX \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{TX}"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"] & TX \end{tikzcd}$$

This means $\mu_X T(\mu_X) = \mu_X \mu_{TX}$ for every set X; that is, the coherence condition $\mu(T\mu) = \mu(\mu T)$ holds, using the notation from Exercise 8 of Section 3. This is an associativity law for expression flattening: an expression of expressions of expressions has one unique flatten into an expression of the symbols.

It is also clear that $e_A \eta_A = 1_A$ for any algebra A, because e_A extends 1_A using the universal property. Taking TX for A, this becomes $\mu_X \eta_{TX} = 1_{TX}$ for every set X; that is, $\mu(\eta T) = 1_T$. Also, since $T(\eta_X) : TX \to TTX$ and $\mu_X : TTX \to TX$ are both homomorphisms, so is $\mu_X T(\eta_X) : TX \to TX$. Because $\mu_X T(\eta_X)$ clearly sends every symbol in X to itself, $\mu_X T(\eta_X) = 1_{TX}$, so that also $\mu(T\eta) = 1_T$.

The statement $\mu(T\mu) = \mu(\mu T)$ mathematically states fact (4) and $\mu(T\eta) = 1_T = \mu(\eta T)$ states fact (5).

All of this leads to the following definition, which enables "expressions" to be natured on objects of an arbitrary category.

DEFINITION

*A **monad** is a triple (T, η, μ) where $T : \mathbf{C} \to \mathbf{C}$ is a functor and $\eta : 1_{\mathbf{C}} \Rightarrow T$ and $\mu : TT \Rightarrow T$ are natural transformations, such that $\mu(T\mu) = \mu(\mu T)$ and $\mu(T\eta) = 1_T = \mu(\eta T)$. η is called the **unit** of the monad, and μ is called its*

operator.

EXAMPLES

1. We have shown above that any variety $\mathcal{V}(S)$ in universal algebra induces a monad (T, η, μ) on **Set** such that $TX = F_S(\Omega, X)$ for every set X. In general, for any takeoff from $\mathcal{V}(S_1)$ to $\mathcal{V}(S_2)$, there is a monad (T, η, μ) on $\mathbf{V}(S_2)$ which sends every $\mathcal{V}(S_2)$-algebra to the universal enveloping $\mathcal{V}(S_1)$-algebra, rasterized as a $\mathcal{V}(S_2)$-algebra. η assigns each $B \in \mathcal{V}(S_2)$ the usual inclusion homomorphism $B \to TB$ given by the universal, and μ assigns each B to the evaluation retraction [see Exercise 4 of Section 3] $r_{TB} : TTB \to TB$ in $\mathcal{V}(S_1)$, induced by the universal enveloping the rasterized $\mathcal{V}(S_1)$-algebra TB.

2. Let M be a fixed monoid. Then M induces a monad (T, η, μ) on **Set** as follows: Define $TX = M \times X$ and for each $f : X \to Y$, assign $T(f) : TX \to TY$ to send $(m, x) \to (m, f(x))$. Then let η and μ be given by the monoid structure of M; that is, $\eta_X : X \to TX$ sends $x \to (1, x)$ and $\mu_X : TTX \to TX$ sends $(m, (n, x)) \to (mn, x)$. The naturality of η and μ is clear, and the coherence conditions [e.g. $\mu(T\mu) = \mu(\mu T)$] follow from the associativity and unit laws of the monoid. This seemingly basic example is the special case of Example 1 where $\mathcal{V}(S)$ is the variety of M-actions.

3. Define a monad (T, η, μ) on **Set** by assigning $TX = \mathcal{P}(X)$ and $T(f) : X \to Y$ the map giving the image of a subset. Then assign $\eta_X : X \to TX$ to send each $x \in X$ to the one-element subset $\{x\}$, and assign $\mu_X : TTX \to TX$ to send each set of subsets of X to their union. It is not hard to show that this defines a monad.

An interesting thing about a monad is that every adjunction yields a monad and vice versa. Throughout this section, we will use the definition of an adjunction in terms of its unit and counit [see Exercise 5 of Section 8]. That is:

I. Functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$

II. Natural transformations $\delta : 1_{\mathbf{D}} \Rightarrow FG, \epsilon : GF \Rightarrow 1_{\mathbf{C}}$

III. $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$

Recall the identities in Exercise 8(c) of Section 3; they will be very useful here. The following result is yielded.

THEOREM 2.9 *Let (G, F, δ, ϵ) be an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit δ and counit ϵ. Then $(FG, \delta, F\epsilon G)$ is a monad on the category $\mathbf{D}$.*

Taking the adjunction given by a takeoff and its universal, for instance, gives us Example 1 above.

Proof of Theorem 2.9. Clearly, $T = FG : \mathbf{D} \to \mathbf{D}$ so that makes sense. $\eta = \delta$ is a natural transformation $1_{\mathbf{D}} \Rightarrow FG$, hence a natural transformation $1_{\mathbf{D}} \Rightarrow T$. Since $\epsilon : GF \Rightarrow 1_{\mathbf{C}}$, $\mu = F\epsilon G : F(GF)G \Rightarrow F1_{\mathbf{C}}G$; that is, $\mu : FGFG \Rightarrow FG$, so μ is a natural transformation $TT \Rightarrow T$. We now prove the coherence conditions:

$$\mu(T\mu) = (F\epsilon G)[(FG)(F\epsilon G)] = (F\epsilon G)(FGF\epsilon G) = F[\epsilon(GF\epsilon)]G$$

$$\mu(\mu T) = (F\epsilon G)[(F\epsilon G)(FG)] = (F\epsilon G)(F\epsilon GFG) = F[\epsilon(\epsilon GF)]G$$

Yet $\epsilon(GF\epsilon) = \epsilon(\epsilon GF)$ by Exercise 8(e) of Section 3, with GF in place of F and G, $1_{\mathbf{C}}$ in place of F' and G' and ϵ in place of ζ and η. Therefore, $\mu(T\mu) = \mu(\mu T)$.

To show that $\mu(T\eta) = 1_T = \mu(\eta T)$:

$$\mu(T\eta) = (F\epsilon G)(FG\delta) = F[(\epsilon G)(G\delta)] = F1_G = 1_{FG} = 1_T$$

$$\mu(\eta T) = (F\epsilon G)(\delta FG) = [(F\epsilon)(\delta F)]G = 1_F G = 1_{FG} = 1_T$$

This completes the proof of the theorem. ■

The Eilenberg-Moore category

You probably suspected that once expressions are defined, there's a general way to give an object "expression assignments" to make it an algebra. There certainly is. Also, there is a way to say a morphism of objects "preserves" the expression assignments. We now make this rigorous.

Recall the evaluation map $e_A : F_S(\Omega, A) \to A$ for $A \in \mathcal{V}(S)$. If $i_A : A \to F_S(\Omega, A)$ is the inclusion map, then $e_A i_A = 1_A$, because e_A extends the set map 1_A. This illustrates that e_A assigns primitive expressions to their symbols, which is genuinely required of an expression evaluation.

Another important thing about e_A is that if you have an expression of expressions in A, evaluating all the inner expressions then evaluating the resulting expression of symbols gives the same result as flattening the expression and then evaluating. This is, in effect, illustrated by the fact that e_A is a homomorphism. In symbols, $e_A e_{F_S(\Omega,A)} = e_A \tilde{e}_A$, where $\tilde{e}_A$ is the homomorphism $F_S(\Omega, F_S(\Omega, A)) \to F_S(\Omega, A)$ extending the set map $i_A e_A : F_S(\Omega, A) \to F_S(\Omega, A)$.

Now consider the evaluation homomorphisms of two algebras, $e_A : F_S(\Omega, A) \to A$ and $e_B : F_S(\Omega, B) \to B$. If $f : A \to B$ is any function of the sets, $i_B f : A \to F_S(\Omega, B)$, has a unique extension to a homomorphism $\tilde{f} : F_S(\Omega, A) \to F_S(\Omega, B)$, and this $\tilde{f}$ makes the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"'] & B \arrow[d, "i_B"] \\ F_S(\Omega, A) \arrow[r, "\tilde{f}"] & F_S(\Omega, B) \end{tikzcd}$$

commutative. We claim that f is a homomorphism if and only if *this* diagram

$$\begin{tikzcd} F_S(\Omega, A) \arrow[r, "\tilde{f}"] \arrow[d, "e_A"'] & F_S(\Omega, B) \arrow[d, "e_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is commutative. For, if the diagram is commutative, f is the result of injectifying the homomorphism $e_B\tilde{f}$ over the quotient map e_A, and is therefore a homomorphism. Conversely, if f is a homomorphism, then for all $\omega \in$

$\Omega(n), a_1, a_2, \ldots, a_n \in A$,

$$e_B\tilde{f}(\overline{\omega}a_1a_2\ldots a_n) = e_B(\overline{\omega}f(a_1)f(a_2)\ldots f(a_n)) = (\omega f(a_1)f(a_2)\ldots f(a_n))$$

$$= e_B(\omega f(a_1)f(a_2)\ldots f(a_n)) = e_B\tilde{f}(\omega a_1a_2\ldots a_n)$$

where the $\overline{\omega}$'s give the unevaluated expression in the free algebra given by the set. It follows that $\ker e_A \subseteq \ker(e_B\tilde{f})$ and that the injectification of $e_B\tilde{f} = ge_A$ for some homomorphism g. Then, $g = g1_A = ge_Ai_A = e_B\tilde{f}i_A = e_Bi_Bf = 1_Bf = f$ so $g = f$ and the statement of the diagram, $e_B\tilde{f} = fe_A$ holds.

To get a better understanding, note that one pair of arrows in the above diagram sends all the symbols in the expression over the map f, then evaluates the resulting expression, whereas the other pair evaluates the expression then sends it over f. The virtue of f being a homomorphism is that there's no difference between those.

We now define the Eilenberg-Moore category.

DEFINITION

Let (T, η, μ) be a monad. An ***algebra*** *for (T, η, μ) is a pair (A, e_A) with $A \in ob(\mathbf{C})$, $e_A : TA \to A$ such that $e_A\eta_A = 1_A$ and $e_A\mu_A = e_AT(e_A)$:*

$$\begin{tikzcd} & TA \arrow[dr, "e_A"] & \\ A \arrow[ur, "\eta_A"] \arrow[rr, "1_A"'] & & A \end{tikzcd} \qquad \begin{tikzcd} TTA \arrow[r, "T(e_A)"] \arrow[d, "\mu_A"'] & TA \arrow[d, "e_A"] \\ TA \arrow[r, "e_A"] & A \end{tikzcd}$$

If (A, e_A) and (B, e_B) are algebras for (T, η, μ), an ***algebra morphism*** *$f : (A, e_A) \to (B, e_B)$ is a morphism $f : A \to B$ in $\mathbf{C}$ satisfying $e_BT(f) = fe_A$:*

$$\begin{tikzcd} TA \arrow[r, "T(f)"] \arrow[d, "e_A"'] & TB \arrow[d, "e_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The ***Eilenberg-Moore category*** *given by the monad is defined to be the category of algebras whose morphisms are algebra morphisms, with the usual composition and identity morphisms. This category is denoted $\mathbf{C}^T$.*

It is immediate that for any $X \in \mathbf{ob}(\mathbf{C})$, (TX, μ_X) is an algebra, and if (A, e_A) is an algebra, then $e_A : (TA, \mu_A) \to (A, e_A)$ is an algebra morphism. Also, for any morphism $f : X \to Y$, $T(f)$ is an algebra morphism $(TX, \mu_X) \to (TY, \mu_Y)$ because of the naturality of μ.

EXAMPLES

1. Let (T, η, μ) be the monad on **Set** given by a variety $\mathcal{V}(S)$. We have seen that a $\mathcal{V}(S)$ algebra A, along with its evaluation homomorphism e_A :

$F_S(\Omega, A) \to A$, is an algebra for (T, η, μ). Conversely, any algebra for the monad is a $\mathcal{V}(S)$ algebra in a clean, deterministic way. Also, algebra morphisms are simply homomorphisms in $\mathbf{V}(S)$, making the Eilenberg-Moore category identifiably $\mathbf{V}(S)$.

2. Let (T, η, μ) be the monad on **Set** with $TX = \mathcal{P}(X)$ as in Example 3. We wish to find its Eilenberg-Moore category. To begin with, an algebra for (T, η, μ) is a set A with a map $e_A : \mathcal{P}(A) \to A$ such that $e_A \eta_A = 1_A$ and $e_A \mu_A = e_A T(e_A)$. Using the definition of η and μ for this monad, this says $e_A(\{a\}) = a$ for $a \in A$ and $e_A(\bigcup S_\alpha) = e_A(\{e_A(S_\alpha)\})$ for subsets S_α of X.

If $a, b \in A$, define $a \leq b$ to mean $e_A(\{a, b\}) = b$. We claim that A is a complete join-semilattice with e_A giving the least upper bound of any set. $e_A(\{a, a\}) = e(\{a\}) = a$, so $a \leq a$, proving reflexivity. If $a \leq b$ and $b \leq a$, then $e_A(\{a, b\})$ is simultaneously a and b, whence $a = b$, proving antisymmetry. Now suppose $a \leq b$ and $b \leq c$. Then $e_A(\{a, b\}) = b$ and $e_A(\{b, c\}) = c$. Consequently,

$$\begin{aligned}
& e_A(\{a, c\}) \\
&= e_A(\{\ e_A(\{a\}),\ e_A(\{b, c\})\ \}) && \text{(because } e_A(\{a\}) = a \text{ and } e_A(\{b, c\}) = c\text{)} \\
&= e_A(\{a\} \cup \{b, c\}) && \text{(because } e_A(\{e_A(S_\alpha)\}) = e_A(\textstyle\bigcup S_\alpha)\text{)} \\
&= e_A(\{a, b, c\}) \\
&= e_A(\{a, b\} \cup \{c\}) \\
&= e_A(\{\ e_A(\{a, b\}),\ e_A(\{c\})\ \}) && \text{(because } e_A(\{e_A(S_\alpha)\}) = e_A(\textstyle\bigcup S_\alpha)\text{)} \\
&= e_A(\{b, c\}) && \text{(because } e_A(\{a, b\}) = b \text{ and } e_A(\{c\}) = c\text{)} \\
&= c.
\end{aligned}$$

Therefore, $e_A(\{a, c\}) = c$ and $a \leq c$, proving transitivity. Hence $(A, \leq)$ is a poset. Now we show that for any subset X, $e_A(X)$ is the least upper bound of X. Suppose $x \in X$; then $e_A(\{x, e_A(X)\}) = e_A(\{\ e_A(\{x\}),\ e_A(X)\ \}) = e_A(\{x\} \cup X) = e_A(X)$, proving that $x \leq e_A(X)$. Therefore $e_A(X)$ is an upper bound of X. Now suppose y is any upper bound of X. If $X \neq \emptyset$, then set theory shows

$$X \cup \{y\} = \bigcup_{x \in X} \{x, y\}$$

So $e_A(\{e_A(X), y\}) = e_A(\{\ e_A(X),\ e_A(\{y\})\ \}) = e_A(X \cup \{y\}) = e_A(\bigcup_{x \in X} \{x, y\}) = e_A(\{e_A(\{x, y\}) \mid x \in X\})$. Since y is an upper bound of X, every $x \in X$ satisfies $x \leq y$ so that $e_A(\{x, y\}) = y$. Furthermore, $e_A(\{e_A(\{x, y\}) \mid x \in X\}) = e_A(\{y \mid x \in X\}) = e_A(\{y\}) = y$. If $X = \emptyset$, then $e_A(\{e_A(X), y\}) = e_A(X \cup \{y\}) = e_A(\{y\}) = y$. Thus $e_A(\{e_A(X), y\}) = y$, which means that $e_A(X) \leq y$ and $e_A(X)$ is the *least* upper bound of X.

This proves that any algebra A for the monad is a complete join-semilattice, which e_A giving the least upper bound. Exercise 1 of Section 10 shows the converse: if A is a complete join-semilattice and $e_A : \mathcal{P}(A) \to A$ gives the least upper bound, then $e_A(\{a\}) = a$ and $e_A(\bigcup S_\alpha) = e_A(\{e_A(S_\alpha)\})$; therefore, A is an algebra for the monad. It is then clear that morphisms are complete join-semilattice homomorphisms, and that the Eilenberg-Moore Category of (T, η, μ) is the category of complete join-semilattices.

We now show one way to get an adjunction from any monad; Exercise 5 shows another way.

THEOREM 2.10 *Let (T, η, μ) be a monad on $\mathbf{C}$, and let $\mathbf{C}^T$ be its Eilenberg-Moore category. Define F, G, δ, ϵ as follows:*

$F : \mathbf{C}^T \to \mathbf{C}$ sends an algebra $(A, e_A) \to A$ and a morphism $h : (A, e_A) \to (B, e_B)$ to the morphism $h : A \to B$ in $\mathbf{C}$.

$G : \mathbf{C} \to \mathbf{C}^T$ sends an object $X \to (TX, \mu_X)$ and a morphism $f : X \to Y$ to the morphism $T(f) : TX \to TY$ of $\mathbf{C}^T$.

For each $X \in ob(\mathbf{C})$, $\delta_X : X \to FGX$ is assigned to be $\eta_X : X \to TX$.

For each $(A, e_A) \in \mathbf{C}^T$, $\epsilon_A : GFA \to A$ is assigned to be $e_A : (TA, \mu_A) \to (A, e_A)$.

Then $\delta : 1_{\mathbf{C}} \Rightarrow FG$ and $\epsilon : GF \Rightarrow 1_{\mathbf{C}^T}$ are natural transformations and G, F are adjoint functors with unit δ and counit ϵ.

Proof of Theorem 2.10. Notice that FG is the functor T, and that δ is the same as η. Therefore, δ is a natural transformation. The naturality of ϵ follows from the definition of a morphism in $\mathbf{C}^T$. It suffices to show that $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$, then we have an adjunction.

Take each $(A, e_A) \in \mathrm{ob}(\mathbf{C}^T)$. Then $[(F\epsilon)(\delta F)]_A = F(\epsilon_A)\delta_{FA} = e_A\eta_{FA} = 1_{FA}$, because FA is simply A without its e_A equipment. Therefore, $(F\epsilon)(\delta F) = 1_F$.

Now take each $X \in \mathrm{ob}(\mathbf{C})$. Then $[(\epsilon G)(G\delta)]_X = \epsilon_{GX} G(\delta_X) = \epsilon_{(TX,\mu_X)} G(\eta_X) = \mu_X T(\eta_X) = [\mu(T\eta)]_X = (1_T)_X = 1_{TX} = 1_{GX}$. Hence $(\epsilon G)(G\delta) = 1_G$. ■

EXERCISES

1. Let $\mathbf{C}$ be the category of complete join-semilattices [Exercise 1 of Section 10], $F : \mathbf{C} \to \mathbf{Set}$ the forgetful functor and $G : \mathbf{Set} \to \mathbf{C}$ the free-object giving functor. Then G and F are adjoint functors. Show that the monad induced by them in Theorem 2.9 is the monad on **Set** in Example 3.

2. A **closure operator** on a poset X is a function $C : X \to X$ satisfying the following laws for all $x, y \in X$:

   Extension: $x \leq C(x)$;

   Idempotence: $C(C(x)) = C(x)$;

   Monotonic Increase: If $x \leq y$, then $C(x) \leq C(y)$.

   For example, if $A \in \mathcal{V}(S)$, "subalgebra generated by" and "congruence relation generated by" are closure operators on $\mathcal{P}(A)$ and $\mathcal{P}(A \times A)$, respectively.

   Show that if X is regarded as a category with at most one morphism in each hom-set [see Example 7 of Section 1], a monad on X is identifiably a closure operator.

3. A comonad on a category $\mathbf{C}$ is a triple (T, η, μ) with $\eta : T \Rightarrow 1_{\mathbf{C}}$, $\mu : T \Rightarrow TT$ natural transformations, such that $(T\mu)\mu = (\mu T)\mu$ and $(T\eta)\mu = 1_T = (\eta T)\mu$ hold.

(a) A comonad on $\mathbf{C}$ is identifiably a monad on $\mathbf{C}^{\text{op}}$.

(b) Let (G, F, δ, ϵ) be an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit δ and counit ϵ. Then $(GF, \epsilon, G\delta F)$ is a comonad on $\mathbf{C}$.

(c) Use part (b) to obtain a comonad on $\mathbf{V}(S)$.

4. (a) If you start with a monad, take its induced adjunction in Theorem 2.10 involving the Eilenberg-Moore category, then take the adjunction's induced monad in Theorem 2.9, show that you have the same monad you started with.

   (b) If you start with an adjunction, take its induced monad [Theorem 2.9] and then the resulting adjunction [Theorem 2.10], then you *don't* necessarily arrive at the same adjunction you started with up to equivalence.

   If the adjunction's "typical" enough that you *would* revisit the same adjunction when doing this, the adjunction is said to be **monadic** [or tripleable].

   (c) Prove that the adjunction given by a takeoff is monadic. [It is best to start off with takeoffs from a variety to the sets.]

5. Let (T, η, μ) be a monad on $\mathbf{C}$, and $\mathbf{C}^T$ be its Eilenberg-Moore category. Define a new category $\mathbf{C}_T$ by $\text{ob}(\mathbf{C}_T) = \text{ob}(\mathbf{C})$ and for $X, Y \in \text{ob}(\mathbf{C}_T)$, $\hom_{\mathbf{C}_T}(X, Y) = \hom_{\mathbf{C}^T}((TX, \mu_X), (TY, \mu_Y))$. Define composition of morphisms and identity morphisms to agree with $\mathbf{C}^T$. Then $\mathbf{C}_T$ is called the **Kleisli category** for the monad.

   (a) Show that there is a bijection between $\hom_{\mathbf{C}_T}(X, Y)$ and $\hom_{\mathbf{C}}(X, TY)$ for all $X, Y \in \text{ob}(\mathbf{C})$. [*Hint*: If $f \in \hom_{\mathbf{C}^T}((TX, \mu_X), (TY, \mu_Y))$, consider $f\eta_X : X \to TY$. The other way around, for each $g : X \to TY$, show that $\mu_Y T(g)$ is in $\mathbf{C}^T$.]

   (b) Define F, G, δ, ϵ as follows:

   $F : \mathbf{C}_T \to \mathbf{C}$ sends each $X \in \text{ob}(\mathbf{C}_T)$ to TX, and each $f \in \hom_{\mathbf{C}_T}(X, Y)$ to f itself as a morphism $TX \to TY$.

   $G : \mathbf{C} \to \mathbf{C}_T$ sends each $X \in \text{ob}(\mathbf{C})$ to X regarded as an object in $\mathbf{C}_T$, and each $f \in \hom_{\mathbf{C}}(X, Y)$ to $T(f) : TX \to TY$ in $\hom_{\mathbf{C}_T}(X, Y)$.

   For each $X \in \text{ob}(\mathbf{C})$, $\delta_X : X \to FGX$ is assigned to be $\eta_X : X \to TX$.

   For each $Y \in \text{ob}(\mathbf{C}_T)$, $\epsilon_Y : GFY \to Y$ is assigned to be $\mu_Y \in \hom_{\mathbf{C}_T}(TY, Y)$.

   Show that δ and ϵ are natural transformations, and that G and F are adjoint functors with unit δ and counit ϵ.

   (c) Prove or disprove: If you use Theorem 2.9 to turn this adjunction into a monad on $\mathbf{C}$, you necessarily get the same monad you started with.

6. Let (G, F, δ, ϵ) be an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit δ and counit ϵ. Then let (T, η, μ) be the monad on $\mathbf{D}$ induced by Theorem 2.9; that is, $(FG, \delta, F\epsilon G)$.

(a) For each $A \in \mathbf{C}$, $(FA, F(\epsilon_A))$ is an algebra for (T, η, μ).

(b) For each $f : A \to A'$ in $\mathbf{C}$, $F(f)$ is an algebra morphism $(FA, F(\epsilon_A)) \to (FA', F(\epsilon_{A'}))$.

(c) Now suppose $(G', F', \delta', \epsilon')$ is the adjunction between $\mathbf{D}^T, \mathbf{D}$ established in Theorem 2.10. Show that there is a unique functor $E : \mathbf{C} \to \mathbf{D}^T$ such that $F = F'E$ and $EG = G'$. Also note that $\delta = \delta'$ because both are equal to η.

(d) Suppose now that $(G', F', \delta', \epsilon')$ is the adjunction between the Kleisli category $\mathbf{D}_T$ and $\mathbf{D}$ in Exercise 5. Define $K : \mathbf{D}_T \to \mathbf{C}$ by $KB = GB$ for $B \in \mathrm{ob}(\mathbf{D}_T)$ [$= \mathrm{ob}(\mathbf{D})$], and $K(f) = \epsilon_{GB'} G(f\eta_B)$ for morphisms $f \in \hom_{\mathbf{D}^T}(TB, TB')$ [$= \hom_{\mathbf{D}_T}(B, B')$]. Then K is the unique functor $\mathbf{D}_T \to \mathbf{C}$ such that $F' = FK$ and $KG' = G$.

7. Suppose (T, η, μ) is a monad on a category $\mathbf{D}$. Define a category $\mathbf{Adj}(\mathbf{D}, T)$ as follows:

   (1) Objects are adjunctions (G, F, δ, ϵ) with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$ where $\mathbf{C}$ is any category, and the induced monad $(FG, \delta, F\epsilon G)$ is the given monad (T, η, μ).

   (2) Morphisms are morphisms of adjunctions [see Exercise 8 of Section 8] which are the identity on $1_{\mathbf{D}}$.

   (a) Verify that this is indeed a category.

   (b) The adjunction involving the Eilenberg-Moore category [Theorem 2.10] is a terminal object. [*Hint*: Use Exercise 6.]

   (c) The adjunction involving the Kleisli category [Exercise 5] is an initial object.