2.10 - Algebraic Categories

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired.

Note: In this section, if A is an object of a structure-based concrete category and B is a subset of A's set, then B is said to be *admitted as a subobject* of A if there exists an object whose set is B such that $\hom(B, A)$ contains the inclusion map $B \hookrightarrow A$ as an embedding. This object is seen to be unique due to the category being structure-based. Likewise, if Φ is an equivalence relation on A, A/Φ is said to be *admitted as a quotient object* of A if there exists an object whose set is A/Φ such that $\hom(A, A/\Phi)$ admits the quotient map $A \to A/\Phi$.

Recall from the previous section that that $\mathbf{V}(S)$ is a structure-based concrete category. However, it is more than just that, and we shall find a somewhat nonconstructive description of $\mathbf{V}(S)$. Some properties of $\mathbf{V}(S)$ that don't hold in an arbitrary structure-based concrete category are:

1. For every set there is a free object.
2. Products exist for any batch of objects.
3. If $f : A \to B$ is any morphism, the category admits the quotient object $A/\ker f$, the subobject $f(A)$ of B and an isomorphism $\sigma : A/\ker f \to B$ such that

$$A \xrightarrow{\pi} A/\ker f \xrightarrow{\sigma} f(A) \xrightarrow{\iota} B$$

is equal to f, with π and ι the canonical maps.

4. If $\Phi \subseteq A \times A$ is an equivalence relation on A which is a subobject of $A \times A$, then the quotient set A/Φ is admitted as a quotient object of A.
5. (Finitary axiom) If $\{A_\alpha\}$ is a batch of subobjects of A which is **directed**, [meaning for all A_α, A_β in the batch there exists A_γ in the batch such that $A_\alpha \subseteq A_\gamma$ and $A_\beta \subseteq A_\gamma$], the union $\bigcup A_\alpha$ is a subobject.

A structure-based concrete category $\mathbf{C}$ satisfying rules 1-5 above is called a **finitary algebraic category**. If $\mathbf{C}$ satisfies rules 1-4 but not necessarily 5, $\mathbf{C}$ is just called an **algebraic category**. See Exercise 1 for an example.

Thus $\mathbf{V}(S)$ is a finitary algebraic category. In this section we show that every finitary algebraic category is of this form.

THEOREM 2.8 *Every finitary algebraic category* $\mathbf{C}$ *is the category* $\mathbf{V}(S)$ *given by some variety* $\mathcal{V}(S)$ *in universal algebra.*

Proof of Theorem 2.8 Let $\mathbf{C}$ be a finitary algebraic category. First note that property 3 implies that in $\mathbf{C}$, all injective morphisms are embeddings, and all surjective morphisms are quotient maps.

We proceed to define a signature Ω by letting $\Omega(n)$, for each $n \geq 0$, be the set F_n underlying the free object given by the n-element set $I_n = \{x_1, x_2, \ldots, x_n\}$. Identify $x_1, x_2, \ldots, x_n$ with their images in the injection map $I_n \to F_n$. We then give each object A an Ω-algebra structure as follows. For each $\omega \in \Omega(n), a_1, a_2, \ldots, a_n \in A$, ω is an element of F_n. Define $(\omega a_1 a_2 \ldots a_n)$ to be $f(\omega)$ where f is the unique morphism $F_n \to A$ extending the set map $x_i \to a_i$ from $I_n \to A$. The reader should take some time to understand the case when $n = 0$.

Now each object of $\mathbf{C}$ is an Ω-algebra. We must prove two things to complete the proof:

(1) The morphisms between two objects of $\mathbf{C}$ are precisely the homomorphisms [i.e., operation-preserving maps] between the algebras.

(2) The class of Ω-algebras is a variety.

It will follow that $\mathbf{C}$ is the category given by a variety.

To prove (1), let $A, B \in \mathbf{C}$. Suppose $f : A \to B$ is a morphism in $\mathbf{C}$. Then take $\omega \in \Omega(n), a_1, a_2, \ldots, a_n \in A$. We must show that $f(\omega a_1 a_2 \ldots a_n) = (\omega f(a_1) f(a_2) \ldots f(a_n))$. Let ψ_A be the map $x_i \to a_i$ from $I_n \to A$, and ψ_B be the map $x_i \to f(a_i)$ from $I_n \to B$. Then ψ_A and ψ_B extend to unique morphisms $\varphi_A : F_n \to A, \varphi_B : F_n \to B$. By definition of the Ω-structures on the objects in $\mathbf{C}$,

$$\varphi_A(\omega) = (\omega a_1 a_2 \ldots a_n)$$

$$\varphi_B(\omega) = (\omega f(a_1) f(a_2) \ldots f(a_n))$$

Now, φ_B is the *unique* morphism $F_n \to B$ sending each $x_i \to f(a_i)$. It is easy to see that $f\varphi_A$ also satisfies this, whence $f\varphi_A = \varphi_B$ by uniqueness. Applying these equal morphisms to ω [the element of F_n] yields the desired statement.

Conversely, suppose $f : A \to B$ is a homomorphism of the Ω-algebras. Now let F_A be the free object given by A, then the set map f extends to a unique morphism $\eta : F_A \to B$ in $\mathbf{C}$, and the identity map $A \to A$ extends to a unique morphism $\epsilon : F_A \to A$. Evidently ϵ is surjective because it is the retraction of a set map. We claim that $\eta = f\epsilon : F_A \to B$. From that it will follow that $\ker \epsilon \subseteq \ker \eta$, and therefore, since ϵ is a quotient map [see the first paragraph of the proof], the injectified result, f, is admitted in $\mathbf{C}$.

Take any $a \in F_A$. By Exercise 4(c), $a \in F_{A'}$ for some finite subset A' of A. Label the elements of A' as $\{a_1, a_2, \ldots, a_n\}$, where $n = |A'|$. Then there is an bijection $F_n \to F_{A'}$ sending $x_i \to a_i$, and a gets mapped to by some $\omega \in F_n = \Omega(n)$. By definition $a = (\omega a_1 a_2 \ldots a_n)$ [the expression in F_A, not its evaluation in A]. Since η sends elements of A to where f sends them,

$$\eta(a) = (\omega f(a_1) f(a_2) \ldots f(a_n))$$

$$\epsilon(a) = (\omega a_1 a_2 \ldots a_n)$$

where the latter expression is the actual evaluation in the algebra A, and the former is the evaluation in the algebra B. The hypothesized statement $f(\omega a_1 a_2 \ldots a_n) = (\omega f(a_1) f(a_2) \ldots f(a_n))$ implies that $\eta(a) = f\epsilon(a)$. This holds for all $a \in F_A$, thus $\eta = f\epsilon$.

Now that we have (1) proved, we must show (2). To show that the class is a variety, by Theorem 1.26 it suffices to show that it is closed under subalgebras, homomorphic images and products, and contains $T(\Omega)$.

To show closure under subalgebras, let A be an algebra in $\mathbf{C}$, and B be a subalgebra of A, in the sense that it is closed under the operations. Then if F_B is the free object given by the set B, the inclusion $B \hookrightarrow A$ extends to a unique homomorphism $f : F_B \to A$. We claim that the image of f is B, so that property 3 implies B is a subobject of A. Clearly each $b \in B$ is f applied to the primitive expression b, so $B \subseteq \text{im } f$. Now suppose $w \in F_B$. Then $w \in F_{B'}$ for some finite subset B' of B. Assume B' is labeled $\{b_1, b_2, \ldots, b_n\}$; then there is an bijection $F_n \to F_{B'}$ sending $x_i \to b_i$, and w gets mapped to by some $\omega \in F_n = \Omega(n)$. By definition of the Ω-structures, w is the expression $(\omega b_1 b_2 \ldots b_n)$. Consequently, $f(w) = (\omega b_1 b_2 \ldots b_n)$, the evaluation of the expression. Since B is a subalgebra of A and $b_1, b_2, \ldots b_n \in B$, $f(w) \in B$ follows, and $\text{im } f \subseteq B$ so that $\text{im } f = B$.

Closure under homomorphic images follows immediately from property 4 [and the fact that $\mathbf{C}$ is structure-based]. Recall the first and foremost definition of a congruence relation given in Section 1.4.

Closure under products follows from property 2. In particular, the empty batch consisting of no objects has a product of $\mathbf{C}$, and hence $T(\Omega)$ is in $\mathbf{C}$. This completes the proof that $\mathbf{C}$ is the category for a variety. ■

At this point it is natural to ask if any of the above 5 properties are redundant. It turns out if $\mathbf{C}$ follows properties 1, 3, 4, 5 but not necessarily 2, it can be proved to be a variety. This is more difficult, though; see Exercise 5.

However, Exercises 1-3 show that none of properties 3, 4, 5 are redundant; there are structure-based concrete categories satisfying any two of those but not the third, as well as satisfying properties 1 and 2.

EXERCISES

1. Here is an example of an algebraic category which is not finitary. Define a **complete join-semilattice** to be a partially ordered set X such that every subset has a least upper bound. For example, the closed interval $[0, 1] \subseteq \mathbb{R}$ is a complete join-semilattice [due to the least upper bound property]. However, $\mathbb{R}$ is *not* a complete join-semilattice, because $\mathbb{Z} \subseteq \mathbb{R}$ has no least upper bound.

   (a) For each subset S of X, let $U(S)$ be the least upper bound of S. Show that $U(\emptyset)$ and $U(X)$ are smallest and largest elements of X, respectively.

   (b) Then prove that for $x \in X$, $U(\{x\}) = x$ and for a set $\{S_\alpha\}$ of subsets, $U(\bigcup S_\alpha) = U(\{U(S_\alpha)\})$.

   (c) Define a **homomorphism** of complete join-semilattices X, Y to be a map $f : X \to Y$ such that for every subset S of X, $f(U(S)) = U(f(S))$. The complete join-semilattices and homomorphisms then form a structure-based concrete category. Show that this is an algebraic category which is not finitary. [*Hint*: For any set Z, $\mathcal{P}(Z)$ is a complete join-semilattice under inclusion. Show that along with $i : Z \to \mathcal{P}(Z)$ sending $i(z) = \{z\}$, it is a free object given by Z.]

2. If X and Y are partially ordered sets, define a map $f : X \to Y$ to be **order-preserving** if $x \leq y$ in X implies $f(x) \leq f(y)$ in Y. Then the category of posets with order-preserving maps satisfies properties 1, 2, 4 and 5 in the definition of a finitary algebraic category, but not property 3. [*Hint*: The free object given by a set X is the poset X where $x \leq y$ means $x = y$.]

3. Define a **torsion-free abelian group** to be an abelian group in which every element except the identity has infinite order. Show that the full subcategory of **Ab** consisting of the torsion-free abelian groups satisfies properties 1, 2, 3 and 5 in the definition of a finitary algebraic category, but not property 4.

4. Let **C** be a finitary algebraic category. Do not use Theorem 2.8 to prove the following statements, because they are used in the proof of that theorem.

   (a) For any set X, let F_X be the free object given by X. If X' is a finite subset of X, explain why there is a unique morphism $F_{X'} \to F_X$ extending the inclusion map $X' \hookrightarrow X$.

   (b) Show that this morphism is injective, and identify elements of $F_{X'}$ with their images in F_X. Then $F_{X'}$ is a subobject of F_X.

   (c) For each $a \in F_X$, there exists a finite subset X' of X such that $a \in F_{X'}$. [This is the primary statement using property 5!]

5. Let **C** be a structure-based concrete category satisfying properties 1, 3, 4 and 5 in the definition of a finitary algebraic category.

   (a) Apply Theorem 2.8 to show that **C** is a class of Ω-algebras for some signature Ω, with all homomorphisms between them.

   (b) Show that the free objects all satisfy the same equational identities in their generating symbols.

   (c) Now show that **C** is the variety of Ω-algebras satisfying those identities.