2.9 - Concrete Categories

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired.

Recall that the category $\mathbf{V}(S)$ has a forgetful functor $\mathbf{V}(S) \to \mathbf{Set}$, which is faithful and continuous. This gives better definitions of "injective," "surjective," "subobject," and "quotient object." One must be cautious though, about the fact that objects of $\mathbf{V}(S)$ are *not* identifiably sets. They do have underlying sets though, and the morphisms are in effect functions of the sets. The functor nature also implies that composition of morphisms and identity morphisms agree with those of the set maps. These kinds of categories have a special name.

DEFINITION

*A **concrete category** is a pair* $(\mathbf{C}, F)$ *with* $\mathbf{C}$ *a category and* $F : \mathbf{C} \to \mathbf{Set}$ *a faithful covariant functor.* F *is called the **forgetful functor** for* $\mathbf{C}$*, and for each* $A \in \mathbf{C}$*,* FA *is called the **underlying set** [or **carrier**] of* A*.*

Since F is faithful, for $f : A \to B$ in $\mathbf{C}$ the map $f \to F(f)$ from $\hom(A, B) \to \hom(FA, FB)$ is injective. Hence $\hom(A, B)$ can be viewed as a subset of $\hom(FA, FB) = FB^{FA}$. One can therefore say a set map $FA \to FB$ is *admitted* as a morphism $A \to B$ or not.

EXAMPLES

1. $\mathbf{V}(S)$ is a concrete category. It has limits for all diagrams which match up with **Set**'s limits because the forgetful functor is continuous. It also has colimits for all diagrams, but those don't match up with the colimits in **Set**.

2. Since there are faithful functors $\mathbf{V}(S){-}\mathbf{sub}, \mathbf{V}(S){-}\mathbf{con} \to \mathbf{V}(S)$ [see Exercise 7 of Section 1], composing them with the forgetful functor $\mathbf{V}(S) \to \mathbf{Set}$ makes $\mathbf{V}(S){-}\mathbf{sub}$ and $\mathbf{V}(S){-}\mathbf{con}$ concrete categories. This can be generalized; if $\mathbf{C}$ is a concrete category and $F : \mathbf{D} \to \mathbf{C}$ is a faithful functor, $\mathbf{D}$ becomes a concrete category.

3. Consider the functor $S : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ sending $(X, Y) \to X \uplus Y$ and (f, g) for $f : X \to X', g : Y \to Y'$ the map $f \uplus g : X \uplus Y \to X' \uplus Y'$ which sends $x \in X$ to $f(x)$ and $y \in Y$ to $g(y)$. It can be shown that S is faithful [this is left to the reader]. This makes $\mathbf{Set} \times \mathbf{Set}$ a concrete category.

4. With parts 2 and 3 combined, the product of any two concrete categories is a concrete category [though one must define it carefully].

If $\mathbb{C}$ is a concrete category with forgetful functor $F : \mathbf{C} \to \mathbf{Set}$ and X is a set, a universal from X to F is called a **free object** given by the set X. If every set X has a free object, then the left adjoint functor of F is called the **free-object giving functor** from $\mathbf{Set} \to \mathbf{C}$.

For example, for $\mathbf{V}(S)$ every set has a free object. But this fails for the functor S in Example 3 [Exercise 4].

Subobjects, quotient objects and Cartesian products

For a concrete category $(\mathbf{C}, F)$, one can conveniently define a morphism f in $\mathbf{C}$ **injective** [**surjective**] if $F(f)$ is injective [surjective]. Then using the fact that F is faithful, it follows that injective morphisms are monic and surjective morphisms are epic. Now call a morphism f **bijective** if it is both injective and surjective. Then all isomorphisms are clearly bijective, but not conversely: a bijective morphism $A \to B$ in $\mathbf{C}$ is only an isomorphism if its inverse is admitted as a morphism $B \to A$.

We now treat objects in concrete categories set theoretically, and introduce the concept of an subobject / quotient object:

DEFINITION

Let $\mathbf{C}$ *be a concrete category,* A, B *objects in* $\mathbf{C}$*. An injective morphism* $\iota : B \to A$ *is called an* ***embedding*** *provided that whenever* $f : C \to A$ *is a morphism such that* $f(C) \subseteq \iota(B)$*, the surjectified result* f_1 *satisfying* $f = \iota f_1$ *is admitted as a morphism* $C \to B$*. In this case,* (B, ι) *is called a* ***subobject*** *of* A*.*

A surjective morphism $\pi : A \to B$ *is called a* ***quotient map*** *provided that whenever* $f : A \to C$ *is a morphism such that* $\ker \pi \subseteq \ker f$*, the injectified result* $\overline{f}$ *satisfying* $f = \overline{f}\pi$ *is admitted as a morphism* $B \to C$*. In this case,* (B, π) *is called a* ***quotient object*** *of* A*.*

At this point, it is convenient to introduce the notion of a **structure-based** concrete category. In such a category, elements of an object's set can be relabeled in any way to get a deterministic result.

DEFINITION

Let $(\mathbf{C}, F)$ *be a concrete category. Then* $(\mathbf{C}, F)$ *is* ***structure-based*** *if for any* $A \in ob(\mathbf{C})$*, set* X *and bijection* $\sigma : FA \to X$*, there is a unique object* $A' \in ob(\mathbf{C})$ *such that* $FA' = X$ *and* σ *is an isomorphism in* $\hom(A, A')$*.*

The definition can be rephrased as follows:

1. For any bijection $\sigma : FA \to X$, there is an object $A' \in \mathrm{ob}(\mathbf{C})$ with $FA' = X$ and σ admitted as an isomorphism in $\hom(A, A')$;
2. Whenever $A, A' \in \mathrm{ob}(\mathbf{C})$ with $FA = FA'$ and 1_A is admitted as an isomorphism in $\hom(A, A')$, then A and A' are the same object.

This is because the second of those conditions states the uniqueness of the object A'. Every concrete category we have mentioned so far is structure-based. Can you come up with an example of a concrete category that isn't structure-based?

Thus in a *structure-based* concrete category, one can define B to be a subobject of A [notation $B \subseteq A$] if $FB \subseteq FA$ and the inclusion map $FB \hookrightarrow FA$ is in $\hom(B, A)$ as an embedding. Any subobject in the sense of the previous definition turns into one of these. In the next section we deal with only structure-based concrete categories, and we shall stick to this definition of a subobject.

Likewise if $\pi : A \to B$ is a quotient map, the codomain can be changed into a unique object such that its underlying set is the quotient set $A/\ker\pi$ and the morphism is the canonical epimorphism. We shall form quotient objects by taking the quotient set from this point.

DEFINITION

Let $(\mathbf{C}, F)$ be a structure-based concrete category, $A_\alpha \in ob(\mathbf{C})$. Then a ***(Cartesian) product*** *of the A_α's is an object A such that:*

1. $FA = \prod FA_\alpha$.

2. The projection maps $p_\alpha : \prod FA_\alpha \to FA_\alpha$ are admitted in $\hom(A, A_\alpha)$.

3. Whenever B is an object in $ob(\mathbf{C})$ and $f_\alpha : B \to A_\alpha$, the coordinate map $f : FB \to \prod FA_\alpha$ satisfying $p_\alpha f = f_\alpha$ is admitted in $\hom(B, A)$.

It is clear that a Cartesian product of objects is a product in the categorical sense. It follows that since $\mathbf{C}$ is structure-based, the object A is unique, and can be referred to as *the* Cartesian product of the A_α's.

The striking controversy is that "product" in a concrete category could refer to either the categorial product or the Cartesian product. In the next section, it will always mean the latter.

EXERCISES

1. Let $(\mathbf{C}, F)$ and $(\mathbf{C}', F')$ be concrete categories. Define a **takeoff** $\mathbf{C} \to \mathbf{C}'$ to be a functor $T : \mathbf{C} \to \mathbf{C}'$ such that $F'T = F$.

   (a) Takeoffs are always faithful.

   (b) The concrete categories with takeoffs as morphisms form a category. [Assume hom-sets are allowed to be proper classes in this occasion.]

   (c) A takeoff of varieties coincides with a takeoff of the concrete categories.

   (d) Informally, what can you say about takeoffs?

2. Let P be the functor $\mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ sending $(X, Y) \to X \times Y$. [See Exercise 7 of Section 4.] Is P faithful? [*Hint*: Consider where P sends $\hom((\mathbb{Z}, \emptyset), (\mathbb{Z}, \emptyset))$.]

3. Define a **congruence relation** Φ on a category $\mathbf{C}$ as follows:

   (1) For each $A, B \in \text{ob}(\mathbf{C})$, an equivalence relation $\Phi_{A,B}$ on $\hom(A, B)$ is equipped.

   (2) Whenever $f, f' \in \hom(A, B)$, $g, g' \in \hom(B, C)$, $f \Phi_{A,B} f'$ and $g \Phi_{B,C} g'$, then $gf \Phi_{A,C} g'f'$.

   (a) Define a new category $\mathbf{C}/\Phi$ by assigning $\text{ob}(\mathbf{C}/\Phi) = \text{ob}(\mathbf{C})$ and $\hom_{\mathbf{C}/\Phi}(A, B) = \hom_{\mathbf{C}}(A, B)/\Phi_{A,B}$. For $\overline{f} \in \hom_{\mathbf{C}/\Phi}(A, B)$, $\overline{g} \in \hom_{\mathbf{C}/\Phi}(B, C)$, set $\overline{g}\overline{f} = \overline{gf}$. Then set $1_A = \overline{1_A} \in \hom_{\mathbf{C}/\Phi}(A, A)$. Explain why this is well-defined, and show that $\mathbf{C}/\Phi$ is a category. It is called the **quotient category** of $\mathbf{C}$ by Φ.

(b) $\pi : \mathbf{C} \to \mathbf{C}/\Phi$ defined by $\pi A = A$, $\pi(f) = \overline{f}$ is a full functor which is bijective on the objects.

(c) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor [assumed covariant]. Define $\Theta_{A,B}$ to be the relation $\{(f, g) \in \hom(A, B)^2 \mid F(f) = F(g)\}$; show that Θ is a congruence relation on $\mathbf{C}$ when defined that way.

(d) If $\pi : \mathbf{C} \to \mathbf{C}/\Theta$ is defined as in part (b), there is a unique faithful functor $\overline{F} : \mathbf{C}/\Theta \to \mathbf{D}$ such that $F = \overline{F}\pi$. [*Hint*: This is similar to Theorem 1.10 in Chapter 1.]

(e) Now explain how to make $\mathbf{C}$ a concrete category given *any* functor $\mathbf{C} \to \mathbf{Set}$.

4. Let S be the functor in Example 3, sending a pair of sets to its disjoint union, and making $\mathbf{Set} \times \mathbf{Set}$ a concrete category. If X is a set, free object given by X would be a pair of sets (A, B) and a function $i : X \to A \uplus B$ such that whenever $f : X \to A' \uplus B'$ is a function, there are unique functions $a : A \to A', b : B \to B'$ such that $f = (a \uplus b)i$. For $X = \emptyset$, obviously $(\emptyset, \emptyset)$ works. However, if $X \neq \emptyset$ show that no such pair of sets exists. [*Hint*: Pick an element of X and consider what parts of the disjoint unions i and f send them to.]

5. If a structure-based concrete category contains at least one nonempty set as an object, then it is not small.