2.8 - Adjunctions

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

We learned universals in Section 5. In this section, we use them to define a functor. Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor such that for every object B in $\mathbf{D}$, there exists a universal (U, u) from B to F.

Now let $\hom(B, F-)$ be the functor $\hom(B, -)F : \mathbf{C} \to \mathbf{Set}$; it sends $A \to \hom_{\mathbf{D}}(B, FA)$ and a morphism $f : A \to A'$ to $\hom(B, F(f)) : \hom(B, FA) \to \hom(B, FA')$ [this makes sense since $F(f) : FA \to FA'$]. We claim that $\hom(B, F-)$ is representable with (U, u) as a representative; to see this, define $\eta_A : \hom(U, A) \to \hom(B, FA)$ sending $h \to F(h)u$. Then clearly $\eta_U(1_U) = u$ and η is a natural transformation from $\hom(U, -)$ to $\hom(B, F-)$; that is, the η_A's are natural in the right variable. Also, by virtue of a universal, for every $A \in \mathrm{ob}(\mathbf{C}), f : B \to FA$ there is a unique morphism $h : U \to A$ such that $f = F(h)u$. This says that for every $A \in \mathrm{ob}(\mathbf{C})$, η_A is a bijection. Therefore, η is a natural isomorphism.

We proceed to define a functor $G : \mathbf{D} \to \mathbf{C}$ as in Example 12 of Section 3. For each $B \in \mathbf{D}$, let (GB, u_B) be any universal from B to F. then the η_A's in the above paragraph are bijections $\hom(GB, A) \to \hom(B, FA)$ which are natural in the right variable. To define $G(f)$ for $f : B \to B'$ in $\mathbf{D}$, note that by the universality of (GB, u_B) there is a unique morphism $\tilde{f} : GB \to GB'$ such that $F(\tilde{f})u_B = u_{B'}f$:

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "u_B"'] & B' \arrow[d, "u_{B'}"] \\
FGB \arrow[r, "F(\tilde{f})"] & FGB'
\end{tikzcd}$$

Set $G(f) = \tilde{f}$. Then $FG(f)u_B = u_{B'}f$ for all $f : B \to B'$ in $\mathbf{D}$. For $f : B \to B', g : B' \to B''$ in $\mathbf{D}$, the commutativity of the squares in

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "u_B"'] & B' \arrow[r, "g"] \arrow[d, "u_{B'}"] & B'' \arrow[d, "u_{B''}"] \\
FGB \arrow[r, "FG(f)"] & FGB' \arrow[r, "FG(g)"] & FGB''
\end{tikzcd}$$

and the functorial property $FG(g)FG(f) = F(G(g)G(f))$ imply that this diagram is commutative:

$$\begin{tikzcd}
B \arrow[r, "gf"] \arrow[d, "u_B"'] & B'' \arrow[d, "u_{B''}"] \\
FGB \arrow[r, "F(G(g)G(f))"] & FGB''
\end{tikzcd}$$

Therefore, $G(g)G(f)$ satisfies the property satisfied by only the morphism $G(gf)$, meaning $G(gf) = G(g)G(f)$. Likewise, $G(1_B) = 1_{GB}$. Therefore, G is a functor.

Now for all $A \in \mathrm{ob}(\mathbf{C}), B \in \mathrm{ob}(\mathbf{D})$, define $\eta_{B,A} : \hom(GB, A) \to \hom(B, FA)$ as above; $\eta_{B,A}(h) = F(h)u_B$. Then we have already seen that for each fixed B, $A \to \eta_{B,A}$ is a natural isomorphism from $\hom(GB, -) \Rightarrow \hom(B, F-)$. We claim that the η's are natural in the left variable now; that is, for each fixed A, $B \to \eta_{B,A}$ is a natural isomorphism from $\hom(G-, A) \Rightarrow \hom(-, FA)$. Well, suppose $f : B' \to B$ in $\mathbf{D}$, then

$$\begin{tikzcd}
\hom(GB, A) \arrow[r, "\eta_{B,A}"] \arrow[d, "\hom(G(f),A)"'] & \hom(B, FA) \arrow[d, "\hom(f,FA)"] \\
\hom(GB', A) \arrow[r, "\eta_{B',A}"] & \hom(B', FA)
\end{tikzcd}$$

is commutative, because for all $h \in \hom(GB, A)$,

$$\hom(f, FA)(\eta_{B,A}(h)) = \eta_{B,A}(h)f = F(h)u_B f$$

$$\eta_{B',A}(\hom(G(f), A)(h)) = \eta_{B',A}(hG(f)) = F(hG(f))u_{B'} = F(h)FG(f)u_{B'}$$

and they are equal because $FG(f)u_{B'} = u_B f$. Thus η is natural in both variables separately. This leads to the following definition.

DEFINITION

*An **adjunction** a triple (G, F, η) where $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ are functors and η assigns each $B \in ob(\mathbf{D}), A \in ob(\mathbf{C})$ a bijection $\eta_{B,A} : \hom(GB, A) \to \hom(B, FA)$ and is natural in both variables as above. In this case, η is called the **adjugant**, G the **left adjoint functor** and F the **right adjoint functor**.*

We have shown that the functor which sends objects in the codomain category to their universals — in particular, the functor in Example 12 of Section 3 — is a left adjoint functor of F. Surprisingly, the converse is true as well:

THEOREM 2.6 *Let (G, F, η) be an adjunction, wth $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$. Then for every $B \in ob(\mathbf{D})$, $(GB, \eta_{B,GB}(1_{GB}))$ is a universal from B to F.*

Proof of Theorem 2.6. Let $u = \eta_{B,GB}(1_{GB}) : B \to FGB$. Then suppose $A \in \mathrm{ob}(\mathbf{C})$ and $f : B \to FA$. We wish to show that there is a unique morphism $h : GB \to A$ such that

$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[dr, "f"'] & FGB \arrow[d, "F(h)"] \\
& FA
\end{tikzcd}$$

is commutative. To show this, we need only show that $\eta_{B,A}$ is precisely the map $h \to F(h)u$ from $\hom(GB, A) \to \hom(B, FA)$, for then the bijectivity of $\eta_{B,A}$ implies that there is a unique h such that $f = F(h)u$, proving the theorem.

For each $h : GB \to A$, the following diagram

$$\begin{tikzcd} \hom(GB, GB) \arrow[r, "\eta_{B,GB}"] \arrow[d, "\hom(GB,h)"'] & \hom(B, FGB) \arrow[d, "\hom(B,F(h))"] \\ \hom(GB, A) \arrow[r, "\eta_{B,A}"] & \hom(B, FA) \end{tikzcd}$$

commutes due to the η's being natural in the right variable. In particular, sending 1_{GB} along each pair of arrows yields:

$$\eta_{B,A}(\hom(GB, h)(1_{GB})) = \eta_{B,A}(h 1_{GB}) = \eta_{B,A}(h)$$

$$\hom(B, F(h))(\eta_{B,GB}(1_{GB})) = \hom(B, F(h))(u) = F(h)u$$

Therefore, $\eta_{B,A}(h) = F(h)u$, and $\eta_{B,A}$ sends each h to $F(h)u$. ■

The fact that a left adjoint functor necessarily sends objects to their universals, makes them unique up to a natural isomorphism.

THEOREM 2.7 *Any two left adjoint functors of F are naturally isomorphic.*

Proof of Theorem 2.7. Let (G, F, η) and (G', F, ζ) be adjunctions with $F : \mathbf{C} \to \mathbf{D}$, $G, G' : \mathbf{D} \to \mathbf{C}$. For $B \in \mathrm{ob}(\mathbf{D})$ assign $u_B = \eta_{B,GB}(1_{GB})$ and $v_B = \zeta_{B,G'B}(1_{G'B})$. Then by Theorem 2.6, (GB, u_B) and $(G'B, v_B)$ are universals from B to F. Therefore, there is a unique isomorphism $\sigma_B : GB \to G'B$ such that $v_B = F(\sigma_B)u_B$. We claim that σ is a natural isomorphism $G \Rightarrow G'$ when defined this way. To show this, we must show that for all $f : B \to B'$,

$$\begin{tikzcd} GB \arrow[r, "\sigma_B"] \arrow[d, "G(f)"'] & G'B \arrow[d, "G'(f)"] \\ GB' \arrow[r, "\sigma_{B'}"] & G'B' \end{tikzcd}$$

is commutative. Applying $\eta_{B,G'B'}$ to both composite arrows,

$$\eta_{B,G'B'}(G'(f)\sigma_B) = F(G'(f)\sigma_B)u_B = FG'(f)F(\sigma_B)u_B = FG'(f)v_B = v_{B'}f$$

$$\eta_{B,G'B'}(\sigma_{B'}G(f)) = F(\sigma_{B'}G(f))u_B = F(\sigma_{B'})FG(f)u_B = F(\sigma_{B'})u_{B'}f = v_{B'}f$$

Whence $G'(f)\sigma_B = \sigma_{B'}G(f)$ because $\eta_{B,G'B'}$ is a bijection. Therefore, σ is a natural isomorphism, and G and G' are naturally isomorphic. ■

Everything we have shown dualizes to universals from a functor to an object. A functor sending objects to those kinds of universals is a right adjoint functor. We leave the verification of this to the reader.

EXERCISES

1. Describe the left adjoint functor of:

   (a) The identity functor on any category.

   (b) A takeoff of varieties.

   (c) The functor **Set** $\to$ **Set** sending every set X to X^Y, with Y a fixed set. [*Hint*: send each X to $Y \times X$.]

   (d) The functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ in Exercise 1 of Section 6.

   (e) The constant functor onto some terminal object in $\mathbf{D}$.

2. Show that for a fixed monoid M, the forgetful functor $M-\mathbf{act} \to \mathbf{Set}$ has both a left adjoint and a right adjoint.

3. Let $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ be functors. Show that the following two are functors from $\mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$.

   (a) $\hom(G-,-)$, sending $(B,A) \to \hom(GB,A)$ and (g,f) with $g : B \to B'$ [that is, $g : B' \to B$ in $\mathbf{D}$] and $f : A \to A'$ the map $\hom(GB,A) \to \hom(GB',A')$ sending $h \to fhG(g)$.

   (b) $\hom(-,F-)$, sending $(B,A) \to \hom(B,FA)$ and (g,f) with $g : B \to B'$ and $f : A \to A'$ the map $\hom(B,FA) \to \hom(B',FA')$ sending $h \to F(f)hg$.

   (c) An adjunction is [identifiably] a triple (G,F,η) with η a natural isomorphism $\hom(G-,-) \Rightarrow \hom(-,F-)$.

4. Suppose (G_1,F_1,η^1) and (G_2,F_2,η^2) are adjunctions with

   $$F_1 : \mathbf{C} \to \mathbf{D}, G_1 : \mathbf{D} \to \mathbf{C}$$

   $$F_2 : \mathbf{D} \to \mathbf{E}, G_2 : \mathbf{E} \to \mathbf{D}$$

   For $B \in \mathbf{E}, A \in \mathbf{C}$, set $\eta_{B,A} = (\eta^2_{B,F_1A})(\eta^1_{G_2B,A})$. Show that (G_1G_2, F_2F_1, η) is an adjunction.

5. Let (G,F,η) be an adjunction.

   (a) For each $B \in \mathrm{ob}(\mathbf{D})$, assign $\delta_B = \eta_{B,GB}(1_{GB}) : B \to FGB$. Then δ is a natural transformation $1_{\mathbf{D}} \Rightarrow FG$. δ is called the **unit** of the adjunction.

   (b) For each $A \in \mathrm{ob}(\mathbf{C})$, assign $\epsilon_A = \eta^{-1}_{FA,A}(1_{FA}) : GFA \to A$. Then ϵ is a natural transformation $GF \Rightarrow 1_{\mathbf{C}}$. ϵ is called the **counit** of the adjunction.

   (c) $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$, using the notation from Exercise 8 of Section 3. These are summarized as the following being the identity natural transformations:

   $$F \xrightarrow{\delta F} FGF \xrightarrow{F\epsilon} F$$

   $$G \xrightarrow{G\delta} GFG \xrightarrow{\epsilon G} G$$

(d) Suppose $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ are any functors, and $\delta : 1_{\mathbf{D}} \Rightarrow FG$ and $\epsilon : GF \Rightarrow 1_{\mathbf{C}}$ are any natural transformations, such that $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$. Show that there is a unique adjugant η making (G, F, η) an adjunction with unit δ and counit ϵ. Thus adjunctions could be defined in terms of their units and counits.

6. Every right adjoint functor is continuous, and every left adjoint functor is cocontinuous.

7. Give necessary and sufficient conditions on an object A in a category $\mathbf{C}$ for $\hom(A, -)$ to have a left adjoint functor. [*Hint*: Exercise 4 of Section 7.]