2.7 - Hom Functors, Yoneda's Lemma and Representability

Nicholas McConnell

(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

A hom functor is a functor $F : \mathbf{C} \to \mathbf{Set}$ defined in a special way. Though they have a seemingly basic definition, they are actually very important in the future sections of this chapter.

Fix $A \in \text{ob}(\mathbf{C})$, and define F as follows:

1. For each $B \in \text{ob}(\mathbf{C})$, assign FB to the set $\hom(A, B)$;

2. For each $f : B \to B'$ in $\mathbf{C}$, define $F(f)$ to be the set map from $\hom(A, B)$ to $\hom(A, B')$ sending $h \to fh$. [This set map is notated $\hom(A, f)$.]

It is feasibly shown that the data above defines a functor from $\mathbf{C}$ to $\mathbf{Set}$. This functor is denoted $\hom(A, -)$ and is called a **covariant hom functor**.

Now let $F : \mathbf{C} \to \mathbf{Set}$ be any functor and let η be a natural transformation $\hom(A, -) \Rightarrow F$. Then for *any* object B in $\mathbf{C}$, η_B is a set map $\hom(A, B) \to FB$. In particular, taking A for B, η_A is a set map $\hom(A, A) \to FA$. Therefore, $a = \eta_A(1_A)$ is some element of FA. We claim that a completely determines the natural transformation η. To see where η_B sends $f : A \to B$, note the commutativity of

$$\begin{array}{ccc} \hom(A, A) & \xrightarrow{\eta_A} & FA \\ {\scriptstyle \hom(A,f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \hom(A, B) & \xrightarrow{\eta_B} & FB \end{array}$$

due to η being a natural transformation. Traveling 1_A along each pair of arrows yields $\eta_B(\hom(A, f)(1_A)) = \eta_B(f1_A) = \eta_B(f)$ and $F(f)(\eta_A(1_A)) = F(f)(a)$. Therefore, $\eta_B(f) = F(f)(a)$, which determines η_B.

It is straightforward to show that any a is possible: if a is an arbitrary element of FA, define $\eta_B : \hom(A, B) \to FB$ by $\eta_B(f) = F(f)(a)$ for each B. Then

$$\begin{array}{ccc} \hom(A, B) & \xrightarrow{\eta_B} & FB \\ {\scriptstyle \hom(A,f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \hom(A, B') & \xrightarrow{\eta_{B'}} & FB' \end{array}$$

is commutative for any $f : B \to B'$, because one direction yields $F(f)\eta_B$ and the other $\eta_{B'} \hom(A, f)$ and they are the same set map:

$F(f)(\eta_B(h)) = F(f)(F(h)(a)) = (F(f)F(h))(a) = F(fh)(a) = \eta_{B'}(fh) = \eta_{B'}(\hom(A, f)(h))$

Therefore, η is a natural transformation $\hom(A, -) \Rightarrow F$. Also, $\eta_A(1_A) = F(1_A)(a) = 1_{FA}(a) = a$. What we have proved is summarized as follows.

LEMMA 2.5 (YONEDA'S LEMMA) *Let* $\mathbf{C}$ *be a category,* $A \in ob(\mathbf{C})$*,* $F : \mathbf{C} \to \mathbf{Set}$ *be a functor. Then for every* $a \in FA$ *there is a unique natural transformation* $\eta : \hom(A, -) \Rightarrow F$ *such that* $a = \eta_A(1_A)$*. This natural transformation is given by* η_B *being the map* $f \to F(f)(a)$ *from* $\hom(A, B) \to FB$ *for each* B*.*

Hopefully this sounds like Exercise 3 of Section 1.10 a bit.

Now let $F : \mathbf{V}(S) \to \mathbf{Set}$ be the forgetful functor. Then let A be the free $\mathcal{V}(S)$-algebra given by a single symbol a, and let η be the natural transformation $\hom(A, -) \Rightarrow F$ satisfying $\eta_A(1_A) = a$. We have $\eta_B(f) = F(f)(a)$ for each $f : A \to B$. Since A is the free algebra given by a, then for every $b \in B$ there is a unique homomorphism $f : A \to B$ such that $f(a) = b$; this implies that η_B is bijective, so that η is actually a natural isomorphism. Hence (A, a) is a special pair for the functor F, which yields the following definition.

DEFINITION

Let $F : \mathbf{C} \to \mathbf{Set}$ *be a functor. If there exists a pair* (A, a) *with* $A \in ob(\mathbf{C}), a \in FA$ *such that the natural transformation* $\hom(A, -) \Rightarrow F$ *induced by Lemma 2.5 is a natural isomorphism,* (A, a) *is called a* ***representative*** *of the functor* F*. If functor which has a representative is said to be* ***representable****.*

EXAMPLES

1. If $F : \mathbf{V}(S) \to \mathbf{Set}$ is the forgetful functor, we have seen that F is representable with $(F_S(\Omega, \{a\}), a)$ as a representative.

2. Let $F : \mathbf{C} \to \mathbf{Set}$ be the constant functor onto the one-element set $\{a\}$. Then a representative of F is (I, a) with I an initial object in $\mathbf{C}$. This is because for every $A \in \mathbf{C}$, $\hom(I, A)$ and FA are both one-element sets, so they come in a unique natural bijection. Hence F is representable if and only if $\mathbf{C}$ has an initial object; in particular, this holds if $\mathbf{C} = \mathbf{V}(S)$.

3. (Quotient algebras) Let A be a $\mathcal{V}(S)$-algebra and Φ a congruence relation on A. Define $F : \mathbf{V}(S) \to \mathbf{Set}$ sending B to the subset $\{h \in \hom(A, B) \mid \Phi \subseteq \ker h\}$ of $\hom(A, B)$. Then clearly for $f : B \to B'$, $\hom(A, f)$ sends elements of FB to elements of FB', so it can be restricted to a set map $F(f) : FB \to FB'$. It is easy to see that this data defines a functor.

Now let $\pi : A \to A/\Phi$ be the canonical epimorphism, then $\pi \in F(A/\Phi)$. We claim that $(A/\Phi, \pi)$ is a representative of F. This is because $\eta_B : \hom(A/\Phi, B) \Rightarrow FB$ sends $h \to F(h)(\pi) = h\pi$. The injectification theorem (1.10) shows that for all $f : A \to B$ such that $\Phi \subseteq \ker f$, there is a unique $h : A/\Phi \to B$ such that $f = h\pi$. This means η_B is actually bijective, so η is a natural isomorphism.

4. (Colimits) Let $D : \mathbf{J} \to \mathbf{C}$ be a fixed diagram of type $\mathbf{J}$ in $\mathbf{C}$. Define $F : \mathbf{C} \to \mathbf{Set}$ sending each object B of $\mathbf{C}$ to the set of cocones $(B, \{\eta_\alpha\})$ from D to B. If $f : B \to B'$ is a morphism in $\mathbf{C}$ and $(B, \{\eta_\alpha\})$ is a cocone from D, clearly $(B', \{f\eta_\alpha\})$ is also a cocone; this induces a set map $F(f) : FB \to FB'$. It is straightforward to show that F is a functor.

Now let $(L, \{\eta_\alpha\})$ be a colimit of D. We claim that L, along with this cocone, is a representative of F. The natural transformation $\eta : \hom(L, -) \Rightarrow F$

assigns each $\eta_B, B \in \mathrm{ob}(\mathbf{C})$ the map $\theta \to F(\theta)(L, \{\eta_\alpha\}) = (B, \{\theta\eta_\alpha\})$ from $\hom(L, B) \to FB$. Recall that the virtue of being a colimit is that for any cocone $(B, \{\zeta_\alpha\})$, there is a unique $\theta : L \to B$ such that $\zeta_\alpha = \theta\eta_\alpha$ for every α; that is, $(B, \{\zeta_\alpha\}) = \eta_B(\theta)$. Therefore η_B is bijective.

Conversely, every representative of F is a colimit of D; see Exercise 5 for proof.

We now fix $A \in \mathrm{ob}(\mathbf{C})$ and define a contravariant functor as follows.

1. For each $B \in \mathrm{ob}(\mathbf{C})$, assign FB to the set $\hom(B, A)$.

2. For each $f : B \to B'$ in $\mathbf{C}$, define $F(f)$ to be the map $\hom(B', A) \to \hom(B, A)$ sending $h \to hf$. [This set map is notated $\hom(f, A)$; be wary that B and B' swap places.]

This functor is denoted $\hom(-, A)$ and is called a **contravariant hom functor**. Another way to view this is that $\hom_{\mathbf{C}}(-, A) = \hom_{\mathbf{C}^{\mathrm{op}}}(A, -)$ which is a covariant functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. Yoneda's Lemma then dualizes to $\hom(-, A)$ as follows:

LEMMA 2.5 (CONTINUED) *Let* $F : \mathbf{C} \to \mathbf{Set}$ *be a* contravariant *functor,* $A \in ob(\mathbf{C})$. *Then for every* $a \in FA$ *there is a unique natural transformation* $\eta : \hom(-, A) \Rightarrow F$ *such that* $a = \eta_A(1_A)$. *This natural transformation is given by* η_B *being the map* $f \to F(f)(a)$ *from* $\hom(B, A) \to FB$ *for each* B.

This immediately follows from the first one when $\mathbf{C}^{\mathrm{op}}$ is used. When the pair (A, a) is special enough for η to be a natural isomorphism, F is said to be **representable** with (A, a) as a representative.

EXAMPLES

1. Let $F : \mathbf{C} \to \mathbf{Set}$ be the contravariant constant functor onto the one-element set $\{a\}$. Then a representative of F is (T, a) with T a terminal object in $\mathbf{C}$. Hence F is representable if and only if $\mathbf{C}$ has a terminal object.

2. (Subalgebras) Let A be a $\mathcal{V}(S)$-algebra and A' a subalgebra on A. Define $F : \mathbf{V}(S) \to \mathbf{Set}$ sending B to the subset $\{h \in \hom(B, A) \mid h(B) \subseteq A'\}$ of $\hom(B, A)$. Then clearly for $f : B \to B'$, $\hom(f, A)$ sends elements of FB' to elements of FB, so it can be restricted to a set map $F(f) : FB' \to FB$. This data defines a contravariant functor.

Now let $\iota : A' \to A$ be the canonical monomorphism, then $\iota \in FA'$. We claim that (A', ι) is a representative of F. This is because $\eta_B : \hom(B, A') \Rightarrow FB$ sends $h \to F(h)(\iota) = \iota h$. The bijectivity of each η_B follows from the surjectification theorem (1.5).

3. (Limits) Let $D : \mathbf{J} \to \mathbf{C}$ be a fixed diagram of type $\mathbf{J}$ in $\mathbf{C}$. Define $F : \mathbf{C} \to \mathbf{Set}$ sending each object B of $\mathbf{C}$ to the set of cones $(B, \{\eta_\alpha\})$ from B to D. Then for $f : B' \to B$ a morphism in $\mathbf{C}$ and $(B, \{\eta_\alpha\})$ a cone to D, so is $(B', \{\eta_\alpha f\})$, inducing a set map $F(f) : FB \to FB'$. This defines a contravariant functor F. The reasoning above shows that a representative of F is a limit of the diagram D.

4. Let $\mathcal{P}$ be the contravariant functor of Example 3 of Section 3. Now let $\mathbb{Z}_2$ be the set $\{0,1\}$, then $\{1\}$ is a subset of $\mathbb{Z}_2$, hence is in $\mathcal{P}(\mathbb{Z}_2)$. We claim that $\mathcal{P}$ is representable with $(\mathbb{Z}_2, \{1\})$ as a representative. The natural transformation $\eta : \hom(-, \mathbb{Z}_2) \Rightarrow \mathcal{P}$ sends a set X to the map $\eta_X : \hom(X, \mathbb{Z}_2) \Rightarrow \mathcal{P}(X)$ sending $f \to F(f)(\{1\}) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$. Since for every subset X' of X, the *characteristic function* $f : X \to \mathbb{Z}_2$ [$f(x) = 1$ for $x \in X'$, 0 for $x \in X - X'$] is the unique element of $\hom(X, \mathbb{Z}_2)$ sent to X' by η_X, η_X is bijective. Therefore, η is a natural isomorphism.

The hom functors have a huge use in the next section.

EXERCISES

1. Use Yoneda's Lemma to show that for every $h : A' \to A$ in $\mathbf{C}$, there is a unique natural transformation $\eta : \hom(A, -) \Rightarrow \hom(A', -)$ such that $h = \eta_A(1_A)$.

2. Let $\mathbf{C}$ be an arbitrary category, and define $H : \mathbf{C}^{\text{op}} \times \mathbf{C} \to \mathbf{Set}$ as follows. For each pair (A, B) of objects of $\mathbf{C}$, $H(A, B) = \hom(A, B)$. To define $H(f)$ for $f : (A, B) \to (A', B')$ in $\mathbf{C}^{\text{op}} \times \mathbf{C}$, note that f is a pair of morphisms $f_1 : A' \to A, f_2 : B \to B'$ of $\mathbf{C}$. Assign $H(f)$ the set map $h \to f_2 h f_1$ from $\hom(A, B)$ to $\hom(A', B')$. Verify that this is a functor. It is called the **two-variable hom functor**.

3. Show that $\hom(A, -)$ is continuous. [See Exercise 5 of Section 6.] Conclude that every [covariant] representable functor is continuous.

4. Let X be a set, $A \in \text{ob}(\mathbf{C})$, and $A' = \coprod_{x \in X} A$ be a copower of A with injections $i_x : A \to A'$, $x \in X$. Then define $u : X \to \hom(A, A')$ sending $x \to i_x$. Show that (A', u) is a universal from X to $\hom(A, -)$.

5. A representative of a functor F is precisely a universal from the one-element set $\{\circ\}$ to F. Conclude that if (A, a) and (A', a') are two representatives of F, there is a unique isomorphism $\sigma : A \to A'$ such that $F(\sigma)(a) = a'$.

6. Let $f : A' \to A, g : B \to B'$ in $\mathbf{C}$. Explain why

$$\begin{array}{ccc} \hom(A,B) & \xrightarrow{\hom(f,B)} & \hom(A',B) \\ {\scriptstyle \hom(A,g)}\downarrow & & \downarrow{\scriptstyle \hom(A',g)} \\ \hom(A,B') & \xrightarrow{\hom(f,B')} & \hom(A',B') \end{array}$$

is a commutative diagram. Use this to show that $F : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}}$ is a contravariant functor when one defines $FA = \hom(A, -)$ and for $f : A' \to A$ in $\mathbf{C}$, $F(f) : \hom(A, -) \Rightarrow \hom(A', -)$ is the natural transformation given by $\eta_B = \hom(f, B) : \hom(A, B) \to \hom(A', B)$.