2.6 - Limits and Colimits

Nicholas McConnell

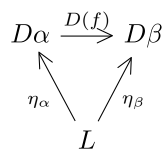
(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired.

Products, pullbacks and difference kernels are all examples of a more general concept called the **limit**. To see this, let $\mathbf{J}$ be an "elementary" category [such as a discrete category], and $D : \mathbf{J} \to \mathbf{C}$ a functor. Then D is called a **diagram of type J** in $\mathbf{C}$.

Define a **cone** to the diagram D to be a natural transformation from a constant functor onto an object L in $\mathbf{C}$ [Example 5 of Section 3] to D. Stated otherwise, it is a pair $(L, \{\eta_\alpha\})$ with $L \in \mathrm{ob}(\mathbf{C})$, $\eta_\alpha : L \to D\alpha$ with $\alpha \in \mathrm{ob}(\mathbf{J})$, such that for every $f \in \hom_{\mathbf{J}}(\alpha, \beta)$ the following diagram is commutative:



Given this data we define

DEFINITION

*Let $D : \mathbf{J} \to \mathbf{C}$ be a diagram of type $\mathbf{J}$ in $\mathbf{C}$. A **limit** of D is defined to be a cone $(L, \{\eta_\alpha\})$ to D such that for every cone $(B, \{\zeta_\alpha\})$ to D, there exists a unique morphism $\theta : B \to L$ such that $\zeta_\alpha = \eta_\alpha \theta$ for all α.*

Again it is routine to show the "uniqueness up to a unique isomorphism": if $(L, \{\eta_\alpha\})$ and $(L', \{\eta'_\alpha\})$ are both limits of D, there is a unique isomorphism $L' \to L$ such that $\eta'_\alpha = \eta_\alpha \sigma$ for all α. For one, this is an immediate consequence of limits being universals for suitably defined categories. See Exercise 1.

EXAMPLES

1. If $\mathbf{J}$ is a discrete category, then its only morphisms are the identity morphisms, and the commutativity of the diagram in the definition of a cone is a tautology. Thus a cone is simply a pair $(L, \{\eta_\alpha\})$ with $L \in \mathrm{ob}(\mathbf{C})$ and $\eta_\alpha : L \to D\alpha$. It is easy to see that a limit of D is simply a product of the $D\alpha$'s [Section 4].

2. Let $\mathbf{J}$ be the category with only two objects α, β such that the only morphisms are the identity morphisms and two morphisms $\alpha \to \beta$. Then a diagram of type $\mathbf{J}$ in $\mathbf{C}$ is a pair of objects $A_\alpha, A_\beta \in \mathrm{ob}(\mathbf{C})$ with two morphisms $f_1, f_2 : A_\alpha \to A_\beta$. A cone is a triple $(L, \eta_\alpha, \eta_\beta)$ such that $\eta_\beta = f_1 \eta_\alpha$ and $\eta_\beta = f_2 \eta_\alpha$, or is identifiably a pair (L, η) with $f_1 \eta = f_2 \eta$. Thus a limit is a difference kernel [or equalizer] [see Exercise 2 of Section 2]. This can be done for more than two morphisms as well.

3. Suppose **J** has three objects, α, β, γ and two nonidentity morphisms, $\alpha \to \gamma$ and $\beta \to \gamma$. Then a diagram of type **J** in **C** is a triple of objects $A_1, A_2, A \in \text{ob}(\mathbf{C})$ with two morphisms $f_1 : A_1 \to A, f_2 : A_2 \to A$. A cone is identifiably a pair (L, η_1, η_2) such that $f_1\eta_1 = f_2\eta_2$ and a limit is therefore a pullback in this case.

One of the things which makes the category $\mathbf{V}(S)$ so special is that it contains all limits. That is, every diagram in $\mathbf{V}(S)$ of any type has a limit, as we now prove. [Such a category is called a **complete category**.]

THEOREM 2.3 *Limits exist in* $\mathbf{V}(S)$ *for any diagram* $D : \mathbf{J} \to \mathbf{V}(S)$.

Proof of Theorem 2.3. Let $D : \mathbf{J} \to \mathbf{V}(S)$ be a diagram. Then define A to be the following subset of the product $\prod_{\alpha \in \text{ob}(\mathbf{J})} D\alpha$:

$$A = \{a \in \prod_{\alpha \in \text{ob}(\mathbf{J})} D\alpha \mid D(f)(a_\alpha) = a_\beta \ \forall f \in \text{hom}_{\mathbf{J}}(\alpha, \beta)\}$$

We claim that A is a subalgebra of the product. Suppose $\omega \in \Omega(0)$, then $(\omega_{\Pi D\alpha}) \in A$ because $D(f)(\omega_{D\alpha}) = (\omega_{D\beta})$ for all $f : \alpha \to \beta$. Now suppose $n \geq 1, \omega \in \Omega(n)$ and $a^1, a^2, \ldots a^n \in A$. Then for all $f : \alpha \to \beta$,

$$D(f)((\omega a^1 a^2 \ldots a^n)_\alpha) = D(f)(\omega a^1_\alpha a^2_\alpha \ldots a^n_\alpha) = (\omega D(f)(a^1_\alpha) D(f)(a^2_\alpha) \ldots D(f)(a^n_\alpha))$$
$$= (\omega a^1_\beta a^2_\beta \ldots a^n_\beta) = (\omega a^1 a^2 \ldots a^n)_\beta$$

Therefore, $(\omega a^1 a^2 \ldots a^n) \in A$, and A is a subalgebra. Now let $\eta_\alpha : A \to D\alpha$ be the *restricted projections* [that is, $p_\alpha(a) = a_\alpha$ for $a \in A$]. Then $(A, \{\eta_\alpha\})$ is a cone to D because the η_α are homomorphisms and for all $f : \alpha \to \beta$ in **J**,

$$\eta_\beta(a) = a_\beta = D(f)(a_\alpha) = D(f)\eta_\alpha(a)$$

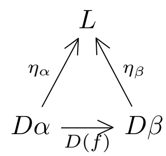
Therefore $\eta_\beta = D(f)\eta_\alpha$.

Now suppose $(B, \{\zeta_\alpha\})$ is any cone to D. Then $\zeta_\alpha : B \to D\alpha$ and one can form the coordinate map $\zeta : B \to \Pi D_\alpha$ satisfying $\zeta(b)_\alpha = \zeta_\alpha(b)$. We claim that im $\zeta \subseteq A$, so that ζ can be surjectified into a homomorphism $\theta : B \to A$. To see this, use the fact that $(B, \{\zeta_\alpha\})$ is a *cone*, and hence

$$D(f)(\zeta(a)_\alpha) = D(f)(\zeta_\alpha(a)) = D(f)\zeta_\alpha(a) = \zeta_\beta(a) = \zeta(a)_\beta$$

Therefore, θ exists and, and obviously $\zeta_\alpha = \eta_\alpha\theta$. Since an element of A is completely determined by where each η_α sends it, θ is unique. Therefore, $(A, \{\eta_\alpha\})$ is a limit. ■

Colimits are the dual of limits, and they are obtained by reversing the arrows. This doesn't mean make the functor $\mathbf{J} \to \mathbf{C}$ contravariant, though [which could be remedied anyway, by changing **J** into $\mathbf{J}^{\text{op}}$]. If $D : \mathbf{J} \to \mathbf{C}$ is a diagram, a **cocone** from D is defined to be a natural transformation from D to a constant functor. In summary, it is a pair $(L, \{\eta_\alpha\})$ with $L \in \mathrm{ob}(\mathbf{C}), \eta_\alpha : D\alpha \to L$ with $\alpha \in \mathrm{ob}(\mathbf{J})$, such that



is commutative for suitable morphisms f in $\mathbf{J}$. This is dual to a cone.

DEFINITION

*Let $D : \mathbf{J} \to \mathbf{C}$ be a diagram of type $\mathbf{J}$ in $\mathbf{C}$. A **colimit** of D is defined to be a cocone $(L, \{\eta_\alpha\})$ from D such that for every cocone $(B, \{\zeta_\alpha\})$ from D, there exists a unique morphism $\theta : L \to B$ such that $\zeta_\alpha = \theta\eta_\alpha$ for all α.*

Once again, it is routine to show uniqueness up to isomorphism of this.

EXAMPLES

1. If $\mathbf{J}$ is a discrete category, then a cocone is simply a pair $(L, \{\eta_\alpha\})$ with $L \in \mathrm{ob}(\mathbf{C})$ and $\eta_\alpha : D\alpha \to L$. The coherence diagram is automatic. It is easy to see that a colimit of D is simply a coproduct of the $D\alpha$'s.
2. Let $\mathbf{J}$ be the category with only two objects α, β such that the only morphisms are the identity morphisms and two morphisms $\alpha \to \beta$. Then a colimit of a diagram is a difference cokernel [or coequalizer] of the two morphisms. This can be done with more than two morphisms as well.
3. Suppose $\mathbf{J}$ has three objects, α, β, γ and two nonidentity morphisms, $\gamma \to \alpha$ and $\gamma \to \beta$. Then a colimit of a diagram is a pushout.

What's quite unbelievable is that $\mathbf{V}(S)$ also contains all colimits! The material covered in the previous chapter can be used to prove this.

THEOREM 2.4 *Colimits exist in $\mathbf{V}(S)$ for any diagram $D : \mathbf{J} \to \mathbf{V}(S)$.*

Proof of Theorem 2.4. Let $D : \mathbf{J} \to \mathbf{V}(S)$ be a diagram. Then let $A = \coprod_{\alpha \in \mathrm{ob}(\mathbf{J})} D\alpha$, with injections $i_\alpha : D\alpha \to A$ for $\alpha \in \mathrm{ob}(\mathbf{J})$. Now, let Θ be the congruence relation on A generated by the following subset of $A \times A$:

$$\{(i_\beta D(f)(a), i_\alpha(a)) \mid f : \alpha \to \beta \text{ in } \mathbf{J}, a \in D\alpha\}$$

Set $L = A/\Theta$, $\pi : A \to L$ the canonical epimorphism and $\eta_\alpha = \pi i_\alpha$ for $\alpha \in \mathrm{ob}(\mathbf{J})$. We claim that $(L, \{\eta_\alpha\})$ is a cocone from D. To show this, we need to show that $\eta_\beta D(f) = \eta_\alpha$ for $f : \alpha \to \beta$ in $\mathbf{J}$. This follows because for all $a \in D_\alpha$, $(i_\beta D(f)(a), i_\alpha(a)) \in \Theta$ by definition, so that $\eta_\beta D(f)(a) = \pi i_\beta D(f)(a) = \pi i_\alpha(a) = \eta_\alpha(a)$. Therefore, $\eta_\beta D(f) = \eta_\alpha$ and $(L, \{\eta_\alpha\})$ is a cocone.

Now suppose $(B, \{\zeta_\alpha\})$ is another cocone from D. Then since each $\zeta_\alpha : D\alpha \to B$ and A is the coproduct of the $D\alpha$'s, there is a unique morphism

$\zeta : A \to B$ such that $\zeta i_\alpha = \zeta_\alpha$ for all α. Whenever $f : \alpha \to \beta \in \mathbf{J}$ and $a \in D\alpha$,

$$\zeta i_\beta D(f)(a) = \zeta_\beta D(f)(a) = \zeta_\alpha(a) = \zeta i_\alpha(a)$$

because the ζ_α's form a cocone; hence $(i_\beta D(f)(a), i_\alpha(a)) \in \ker \zeta$. Since the congruence relation Θ is generated by pairs of that form, $\Theta \subseteq \ker \zeta$, and ζ can be injectified [Theorem 1.10] to a morphism $\theta : L \to B$ satisfying $\zeta = \theta\pi$.

Furthermore, $\zeta_\alpha = \zeta i_\alpha = \theta\pi i_\alpha = \theta\eta_\alpha$ for all α.

Since any homomorphism θ' satisfying $\zeta_\alpha = \theta'\eta_\alpha$ agrees with θ on all elements of images of the η_α, but they generate L, θ is unique, completing the proof. ■

EXERCISES

1. Let $\mathbf{J}, \mathbf{C}$ be categories, and $\mathbf{C}^{\mathbf{J}}$ the functor category. Define the **diagonal functor** $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ by sending each $A \in \mathrm{ob}(\mathbf{C})$ to the constant functor onto A. For $f : A \to B$ in $\mathbf{C}$, $\Delta(f)$ is the natural transformation $\eta : \Delta A \Rightarrow \Delta B$ with $\eta_\alpha = f$ for all α. Show that a limit of a diagram D is a universal from Δ to the object D of $\mathbf{C}^{\mathbf{J}}$, and that a colimit is a universal from D to Δ.

2. Suppose $\iota \in \mathrm{ob}(\mathbf{J})$ is a initial object. If $D : \mathbf{J} \to \mathbf{C}$ is a diagram, then $(D\iota, \{\eta_\alpha\})$ is a limit of D, where η_α is the result of applying D to the unique morphism $\iota \to \alpha$ in $\mathbf{J}$. Dualize.

3. Show that any category with all products [including the terminal object] and equalizers has all limits as follows. Let $D : \mathbf{J} \to \mathbf{C}$ be a diagram. Now let $A = \prod_{\alpha \in \mathrm{ob}(\mathbf{J})} D\alpha$ and $P = \prod_{f \in \hom_{\mathbf{J}}(\alpha,\beta)} D\beta$, where the latter product is taken over all morphisms in $\mathbf{J}$. Denote the projections from A as $p^1_\alpha : A \to D\alpha$ and the projections from P as $p^2_f : P \to D\beta$, $f \in \hom(\alpha, \beta)$.

   (a) Show that there is a unique morphism $\varphi : A \to P$ such that $p^2_f \varphi = p^1_\beta$ for $f \in \hom(\alpha, \beta)$. [*Hint*: If you need a hint, think about how P is defined.]

   (b) Show that there is also a unique $\psi : A \to P$ such that $p^2_f \psi = D(f)p^1_\alpha$ for $f \in \hom(\alpha, \beta)$.

   (c) Now let $\epsilon : L \to A$ be an equalizer of φ and ψ; show that $(L, \{p^1_\alpha \epsilon\})$ is a limit of D.

   (d) In a variety $\mathbf{V}(S)$ in universal algebra, recall that products are direct products, and the equalizer of $f, g : A_1 \to A_2$ is the canonical monomorphism from the subalgebra $\{a \in A_1 \mid f(a) = g(a)\}$. Use this to find limits in $\mathbf{V}(S)$. Are they really different from Theorem 2.3?

4. Let $f_i : A_i \to B$ be morphisms in $\mathbf{C}$ for $i = 1, 2$. Then let $g_i : C \to A_i$, $i = 1, 2$ be a pullback of f_1 and f_2. Prove that if f_1 is monic then so is g_2.

5. A functor $F : \mathbf{C} \to \mathbf{D}$ is **continuous** if it *preserves limits*: Whenever $D : \mathbf{J} \to \mathbf{C}$ is a diagram and $(A, \{\eta_\alpha\})$ is a limit of D, then $(FA, \{F(\eta_\alpha)\})$ is a limit of FD. A **cocontinuous** functor is defined likewise, but for colimits.

   Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties.

   (a) The functor $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$ given by Example 1 of Section 3, is continuous. [*Hint*: Theorem 2.3 shows how to construct the limit. What does the construction depend on?]

   (b) The functor $G : \mathbf{V}(S_2) \to \mathbf{V}(S_1)$ given by Example 12 of Section 3, is cocontinuous. [*Hint*: This is a variation of Exercise 14 of Section 1.11.]

   (c) If $\mathbf{C}$ is a complete category, then any functor $F : \mathbf{C} \to \mathbf{D}$ which preserves products [including the terminal object] and equalizers is continuous. Dualize.