2.5 - Universals

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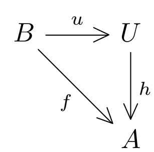
(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

The universals learned in Section 1.11 can be generalized to any functor of categories. Firstly, if $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is a takeoff of varieties, recall what a universal enveloping $B \in \mathcal{V}(S_2)$ is: it consists of a pair (U, u) with $U \in \mathcal{V}(S_1)$ and $u : B \to U$ an Ω_2-homomorphism, such that whenever (A, f) is another such pair, there is a unique Ω_1-homomorphism $h : U \to A$ such that

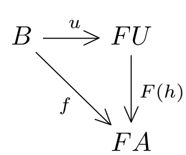


is commutative.

This leads to the following definition. Exercise care in the fact that the statement $f = hu$ treats U, A and h as they are in $\mathbf{V}(S_2)$ when they are virtuously in $\mathbf{V}(S_1)$.

DEFINITION

*Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, $B \in ob(\mathbf{D})$. A **universal from** B **to** F is a pair (U, u) with $U \in ob(\mathbf{C})$ and $u \in \hom_{\mathbf{D}}(B, FU)$ such that whenever (A, f) is another pair with $A \in ob(\mathbf{C})$ and $f \in \hom_{\mathbf{D}}(B, FA)$ there exists a unique $h \in \hom_{\mathbf{C}}(U, A)$ such that the diagram*



*is commutative. U is called the **universal object** and u is called the **universal map**.*

EXAMPLES

1. A takeoff of varieties becomes a functor, and the definition of a universal for that functor coincides with the universal learned in Section 1.11.

In the special case where T is the unique takeoff from $\mathcal{V}(S)$ to the variety of sets, $F : \mathbf{V}(S) \to \mathbf{Set}$ is the forgetful functor, and a universal from a set X to F is $F_S(\Omega, X)$, the free $\mathcal{V}(S)$-algebra given by X.

2. Let **Dom** and **Field** be the categories of integral domains and fields, respectively. Then let $\mathbf{Dom}^m$ be the subcategory of **Dom** keeping all the objects but only the *monomorphisms*. Since every field is an integral domain and every homomorphism of fields is injective, one can form a functor $F : \mathbf{Field} \to \mathbf{Dom}^m$ sending every field and morphism to itself. For any integral domain R, let K

be the field of quotients of R, with injection $i : R \to K$. Then (K, i) is easily seen to be a universal from R to F.

3. Let $\mathbf{C}$ be any category and $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ be the diagonal functor given in Example 14 of Section 3. Then a universal from $(B_1, B_2) \in \text{ob}(\mathbf{C} \times \mathbf{C})$ to Δ is a coproduct of B_1 and B_2. To see this, the universal takes the form (U, u) with $u : (B_1, B_2) \to \Delta U$, that is, [since $\Delta U = (U, U)$], u is a pair of morphisms $u_1 : B_1 \to U, u_2 : B_2 \to U$. The additional property is satisfied that whenever $f : (B_1, B_2) \to \Delta A$, that is, f is a pair of morphisms, $f_1 : B_1 \to A, f_2 : B_2 \to A$, there is a unique morphism $h : B \to A$ such that $f = \Delta(h)u$. Since $\Delta(h) = (h, h)$, this says the same thing as $f_1 = hu_1, f_2 = hu_2$. Therefore, (U, u_1, u_2) is a coproduct of B_1 and B_2.

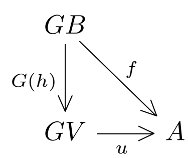
This generalizes to coproducts of more than two objects.

Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor and $B \in \text{ob}(\mathbf{D})$. The proof of Theorem 1.28 applies here, showing that if (U, u) and (U', u') are both universals from B to F, there exists a unique isomorphism $\sigma : U \to U'$ such that $i' = F(\sigma)i$. Thus universals are unique up to a unique isomorphism. There is also a "composition law" for universals; see Exercise 1.

As expected, there is a dual to the definition obtained by reversing the arrows:

DEFINITION

*Let $G : \mathbf{D} \to \mathbf{C}$ be a functor, $A \in ob(\mathbf{C})$. A **universal from** G **to** A is a pair (V, v) with $V \in ob(\mathbf{D})$ and $v \in \hom_{\mathbf{C}}(GV, A)$ such that whenever (B, f) is another pair with $B \in ob(\mathbf{D})$ and $f \in \hom_{\mathbf{C}}(GB, A)$ there exists a unique $h \in \hom_{\mathbf{D}}(B, V)$ such that the diagram*



*is commutative. V is called the **universal object** and v is called the **universal map**.*

EXAMPLES

1. Let $\mathcal{V}(S)$ be a variety and $G : \mathbf{Set} \to \mathbf{V}(S)$ be the free-algebra functor [Example 12 of Section 3]. If $A \in \mathcal{V}(S)$, let V be the *set* A. By virtue of a free algebra, the identity map $V \to A$ [they are the same set, but the codomain is regarded as an algebra] extends to the **evaluation homomorphism** $v : F_S(\Omega, V) \to A$. We claim that (V, v) is a universal from G to A. To see this, let (B, f) be another pair with B a set and $f : GB \to A$ a homomorphism. Then, composing with the inclusion $B \to GB$ yields a unique set map $h : B \to V$ [V is the set A]. Checking on symbols shows that $uG(h) = f$ and h is unique for this property.

This generalizes to the functor $\mathbf{V}(S_2) \to \mathbf{V}(S_1)$ induced by a takeoff $\mathcal{V}(S_1) \to \mathcal{V}(S_2)$ in Example 12 of Section 3. We leave it to the reader to carry out the details.

2. Let M be a fixed monoid and $G : M-\mathbf{act} \to \mathbf{Set}$ be the forgetful functor. If X is a set, we define the **power action** as follows: X^M is the set of all functions from the monoid M to X, and for $\varphi \in X^M$ and $n \in M$, $n\varphi$ is the map $m \to \varphi(mn)$ from $M \to X$. It is straightforward that this makes X^M a power action. Furthermore, the *projection* $p : X^M \to X$ sending $\varphi \to \varphi(1)$ can be considered. We claim that (X^M, p) is a universal from G to X. Thus this functor possesses both kinds of universals.

Suppose Y is any M-action and $f : Y \to X$ is a set map. Then define $h : Y \to X^M$ by assigning $h(y)$ to the map $m \to f(my)$ from $M \to X$. Thus $h(y)(m) = f(my)$. We need to show three things:

(i) h is a homomorphism;

(ii) $ph = f$ as maps $Y \to X$;

(iii) h is unique for properties (i) and (ii).

To show (i), note that for $n \in M$, $h(ny)$ is the map $m \to f(mny)$. On the other hand, $nh(y)$ — by definition of the power action — is the map from $m \to h(y)(mn) = f((mn)y) = f(mny)$. Therefore, $h(ny)$ an $nh(y)$ are equal, so that h is a homomorphism.

(ii) is easy to show because for all $y \in Y$, $ph(y) = p(h(y)) = h(y)(1) = f(1y) = f(y)$. To show (iii), suppose $h' : Y \to X^M$ is also a homomorphism satisfying $ph' = f$. Then for all $y \in Y, m \in M$,

$$h'(y)(m) = h'(y)(1m) = mh'(y)(1) = h'(my)(1) = p(h'(my)) = ph'(my) = f(my)$$

Therefore $h'(y)$ is necessarily the map $m \to f(my)$ for all $y \in Y$, so that $h' = h$ and h is uniquely determined.

EXERCISES

1. Let $F_1 : \mathbf{C} \to \mathbf{D}, F_2 : \mathbf{D} \to \mathbf{E}$ be functors, and $B \in \text{ob}(\mathbf{E})$. If (U_2, u_2) is a universal from B to F_2 and (U_1, u_1) is a universal from U_2 to F_1, prove that $(U_1, F_2(u_1)u_2)$ is a universal from B to F_2F_1.

2. Let $\mathbf{C}$ be any category and $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ be the diagonal functor. Then a universal from Δ to $(B_1, B_2) \in \text{ob}(\mathbf{C} \times \mathbf{C})$ is a product of B_1 and B_2.

3. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, $B \in \text{ob}(\mathbf{D})$. Define a category $\mathbf{D}(B, F)$ as follows: the objects of $\mathbf{D}(B, F)$ are the pairs of the form (A, f) with $A \in \text{ob}(\mathbf{C})$ and $f : B \to FA$. If $(A_1, f_1), (A_2, f_2)$ are objects, a morphism $(A_1, f_1) \to (A_2, f_2)$ in $\mathbf{D}(B, F)$ is an arrow $g : A_1 \to A_2$ such that $f_2 = F(g)f_1$. Verify that this data forms a category, and that a universal from B to F is an initial object of $\mathbf{D}(B, F)$. Dualize.