2.4 - Products and Coproducts

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Objects in a category can be combined in interesting ways. Two of the convenient operators combining them are products and coproducts. To see what they are like, consider $\mathcal{V}(S)$ algebras.

If the A_α are $\mathcal{V}(S)$ algebras, one can take $A = \Pi A_\alpha$, along with the projection homomorphisms $p_\alpha : A \to A_\alpha$ from the product.

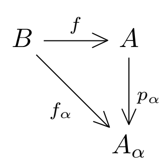
Now suppose B is a Ω-algebra and $f_\alpha : B \to A_\alpha$ is a homomorphism for each α. Define $f : B \to A$ so that $f(b)_\alpha = f_\alpha(b)$. That determines $f(b)$ for each b, and it is seen that f is the only homomorphism $B \to A$ such that $f_\alpha = p_\alpha f$ for all α. This illustrates a product in terms of purely homomorphisms:

Whenever $f_\alpha : B \to A_\alpha$ is a homomorphism for each α, there is a unique homomorphism $f : B \to \Pi A_\alpha$ such that $f_\alpha = p_\alpha f$ for all α.

This property leads to the following definition in category theory.

DEFINITION

*Let $\{A_\alpha\}$ be a batch of objects in a category $\mathbf{C}$ [with possible repetitions]. A **product** of the A_α's is a pair $(A, \{p_\alpha\})$ with $A \in ob(\mathbf{C})$ and $p_\alpha : A \to A_\alpha$ for each α, such that whenever $B \in ob(\mathbf{C})$ and $f_\alpha : B \to A_\alpha$ for each α, there is a unique morphism $f : B \to A$ such that for all α we have $p_\alpha f = f_\alpha$; in other words,*



is commutative.

The fact that the definition says "*a* product," rather than "*the* product," can be remedied, as Exercise 1 shows that products are unique up to a unique isomorphism.

EXAMPLES

1. We have just shown that products in $\mathbf{V}(S)$ coincide with the product of algebras in Chapter 1, Section 2.

2. If $\{(A_\alpha, B_\alpha)\}$ are objects in $\mathbf{V}(S)-\mathbf{sub}$, notice that with each B_α a subalgebra of A_α, ΠB_α is a subalgebra of ΠA_α. We claim that $(\Pi A_\alpha, \Pi B_\alpha)$, along with the usual projections $p_\alpha : \Pi A_\alpha \to A_\alpha$, is a product of the objects in $\mathbf{V}(S)-\mathbf{sub}$. To begin with, the p_α's are admitted by the category since p_α sends elements of ΠB_α to elements of B_α. Now suppose $f_\alpha : (C, C_1) \to (A_\alpha, B_\alpha)$ are morphisms. This requires that each f_α is a homomorphism $C \to A_\alpha$ satisfying

$f_\alpha(C_1) \subseteq B_\alpha$. With that, if $f : C \to A_\alpha$ is the coordinate map [$p_\alpha f = f_\alpha$ for each α], $f(C_1) \subseteq \Pi B_\alpha$. This means f is a morphism $(C, C_1) \to (\Pi A_\alpha, \Pi B_\alpha)$, and is clearly the only one satisfying $p_\alpha f = f_\alpha$. This proves our claim.

3. Let $\{(A_\alpha, \Phi_\alpha)\}$ be objects in $\mathbf{V}(S)-\mathbf{con}$. For $a, b \in \Pi A_\alpha$, define $a\Phi b$ if $a_\alpha \Phi_\alpha b_\alpha$ for all α. Then this is a congruence relation on ΠA_α, and an argument similar to the one above shows that $(\Pi A_\alpha, \Phi)$ is a product of the objects in $\mathbf{V}(S)-\mathbf{con}$.

4. Products in **Cat** are product categories with the projection functors.

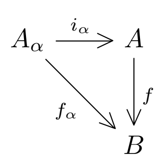
5. If $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ are varieties, one can form a new variety $\mathcal{V}(S_3)$ taking disjoint unions of operators and identities. That is, $\Omega_3(n) = \Omega_1(n) \uplus \Omega_2(n)$ for $n \geq 0$ and $S_3 = S_1 \uplus S_2$. Then a $\mathcal{V}(S_3)$ algebra is precisely a set with both a $\mathcal{V}(S_1)$ structure and a $\mathcal{V}(S_2)$ structure which are independent of one another. Takeoffs $\mathcal{V}(S_3) \to \mathcal{V}(S_1), \mathcal{V}(S_2)$ can be formed, each dropping one of the structures, and $\mathcal{V}(S_3)$ is a product of $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ in the category **Var**.

6. What does it mean for an object T to be a product of the empty batch $\{\}$? Well, there are no p_α's involved in this case, and whenever $B \in \mathrm{ob}(\mathbf{C})$ [there are no f_α's involved], there is a unique morphism $f : B \to T$ [no diagram commutativity is needed]. Stated otherwise, for all $B \in \mathrm{ob}(\mathbf{C})$, $\hom(B, T)$ consists of a single element; in other words, T is a terminal object.

Coproducts are basically the dual of products, and in fact, we have already started them in Section 9 of Chapter 1. They carry over to category theory.

DEFINITION

*Let $\{A_\alpha\}$ be a batch of objects in a category $\mathbf{C}$ [with possible repetitions]. A **coproduct** of the A_α's is a pair $(A, \{i_\alpha\})$ with $A \in ob(\mathbf{C})$ and $i_\alpha : A_\alpha \to A$ for each α, such that whenever $B \in ob(\mathbf{C})$ and $f_\alpha : A_\alpha \to B$ for each α, there is a unique morphism $f : A \to B$ such that for all α we have $f i_\alpha = f_\alpha$; in other words,*



is commutative.

Coproducts are basically products in the opposite category $\mathbf{C}^{\mathrm{op}}$. Coproducts in $\mathbf{V}(S)$ coincide with the definition of a coproduct in Section 9 of Chapter 1. There, we proved that coproducts always exist in $\mathbf{V}(S)$, and here we shall use the proof to derive a subtle and interesting explanation on how to find them.

1. Suppose you are given a batch $\{A_\alpha\}$. Let F be the free algebra given by the set $\uplus A_\alpha$ with *set map* $i : \uplus A_\alpha \to F$. Then form $j_\alpha : A_\alpha \to F$ for each α by composing i with each injection $A_\alpha \to \uplus A_\alpha$.

2. To make each j_α a homomorphism, identify any expression in F whose symbols come from a single A_α with its value given by A_α. That is, identify

$(\omega j_\alpha(a_1)j_\alpha(a_2)\ldots j_\alpha(a_n))$ with $j_\alpha(\omega a_1 a_2 \ldots a_n)$. To do this, find the congruence relation Θ on F generated by those pairs and let $\pi : F \to F/\Theta$ be the canonical epimorphism. *Make no more identifications than that* or it won't work.

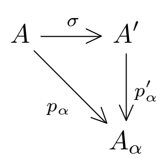
3. Then each πj_α is a homomorphism and $(F/\Theta, \pi j_\alpha)$ is a coproduct of the A_α's.

To make a long story short, the coproduct of algebras consists of expressions whose symbols are in all the algebras, such that the identities in S are satisfied, and any expression with its symbols in a single algebra is identified with the value the algebra gives it. You can give each algebra a different color to see this easily. Operator symbols and parentheses have no color.

In the case of groups, this precise procedure gives the familiar free product on groups; same for monoids. Also, it gives R-modules their direct sum, and commutative rings their *tensor product* [to be learned later].

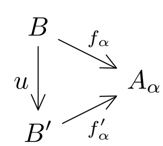
EXERCISES

1. Let $\{A_\alpha\}$ be a batch of objects in a category $\mathbf{C}$. If $(A, \{p_\alpha\})$ and $(A', \{p'_\alpha\})$ are both products of the A_α's in $\mathbf{C}$, there is a unique isomorphism $\sigma : A \to A'$ such that for all indices α the diagram



is commutative. Dualize.

2. Let $\{A_\alpha\}$ be a batch of objects in a category $\mathbf{C}$. Define $\mathbf{C}/\{A_\alpha\}$ as follows: $\text{ob}(\mathbf{C}/\{A_\alpha\})$ is the class of pairs of the form $(B, \{f_\alpha\})$ where $B \in \text{ob}(\mathbf{C})$ and $f_\alpha : B \to A_\alpha$ for each α, and $\text{hom}((B, \{f_\alpha\}), (B', \{f'_\alpha\}))$ is the set of morphisms $u : B \to B'$ such that for every α



is commutative. Define composition of morphisms and identity morphisms as in $\mathbf{C}$. Verify that this data form a category, and that a product of the A_α's is a terminal object in $\mathbf{C}/\{A_\alpha\}$. Dualize.

3. If T is a terminal object of category $\mathbf{C}$, show that A is a product of T and A and determine the projection maps.

4. If I is an initial object of a category $\mathbf{C}$, show that A is a coproduct of I and A.

5. If $\mathbf{C}$ is a category given by a preorder [Example 7 of Section 1], describe products of objects in $\mathbf{C}$.

6. (a) Suppose **C** is a category in which any two objects in **C** have a product in **C**. [Such a category is called a **category with finite products**.] Show that any finite batch of objects in **C** has a product in **C**.

   (b) Give an example of a category **C** with finite products having an infinite batch of objects with no product.

7. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be morphisms in a category **C**. Suppose (A, p_1, p_2) is a product of A_1 and A_2 and (B, q_1, q_2) is a product of B_1 and B_2.

   (a) There is a unique morphism $f : A \to B$ such that the two rectangles in

$$\begin{array}{ccccc} A_1 & \xleftarrow{p_1} & A & \xrightarrow{p_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xleftarrow{q_1} & B & \xrightarrow{q_2} & B_2 \end{array}$$

   are commutative.

   (c) Suppose **C** is a category with finite products. Define $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ as follows: Assign each object (A_1, A_2) to a product of A_1 and A_2, and each morphism (f_1, f_2) to the morphism f given in part (a). Verify that F is a functor. It is called the *product-giving functor* for **C**.

   (d) Any two functors defined in the way of (c) are naturally isomorphic.