2.3 - Functors and Natural Transformations

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Functors are structure-preserving maps of categories. However, since categories are more than just classes of objects, it appears that we haven't really seen a map of categories before. However, we have; consider takeoffs from Section 1.11, but this time regard $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ as categories.

A takeoff gives every $\mathcal{V}(S_1)$-algebra a $\mathcal{V}(S_2)$-algebra structure in such a way that all $\mathcal{V}(S_1)$-homomorphisms preserve the $\mathcal{V}(S_2)$-structure. One can think about this as assigning a $\mathcal{V}(S_1)$-algebra a $\mathcal{V}(S_2)$-algebra and every homomorphism of $\mathcal{V}(S_1)$-algebras a homomorphism of the corresponding $\mathcal{V}(S_2)$-algebras. The composition of maps and identity maps are obviously preserved by this.

Abstracting the above information, we have the following definition:

DEFINITION

If $\mathbf{C}$ *and* $\mathbf{D}$ *are categories, a* ***[covariant] functor*** *from* $\mathbf{C}$ *to* $\mathbf{D}$ *is a mathematical object* F *such that:*

(1) *For each* $A \in ob(\mathbf{C})$, FA *is some object of* $\mathbf{D}$.

(2) *For each* $f : A \to B$ *in* $\mathbf{C}$, $F(f)$ *is some morphism* $FA \to FB$ *of* $\mathbf{D}$.

(3) $F(gf) = F(g)F(f)$ *whenever* gf *is defined in* $\mathbf{C}$.

(4) $F(1_A) = 1_{FA}$ *for all* $A \in ob(\mathbf{C})$.

EXAMPLES

1. A takeoff $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ of varieties becomes a functor $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$ sending every $\mathcal{V}(S_1)$-algebra to itself as a $\mathcal{V}(S_2)$-algebra with the derived structure, and every homomorphism of $\mathcal{V}(S_1)$-algebras to itself. In particular, for any variety $\mathcal{V}(S)$, we have the **forgetful functor** $F : \mathbf{V}(S) \to \mathbf{Set}$ sending every algebra to its underlying set, and every homomorphism to itself as a set map.

2. If $\mathbf{M}$ and $\mathbf{N}$ are monoids viewed as one-object categories [Example 6 of Section 1], a functor $\mathbf{M} \to \mathbf{N}$ is a monoid homomorphism. If $\mathbf{S}$ and $\mathbf{T}$ are preordered sets viewed as categories with at most one morphism in every hom set [Example 7 of Section 1], a functor $\mathbf{S} \to \mathbf{T}$ is an order-preserving map.

3. Let **Poset** be the category consisting of partially ordered sets and order-preserving maps. Evidently a homomorphism $f : A \to B$ of $\mathcal{V}(S)$-algebras induces an order-preserving map from Sub A to Sub B. Hence we have a functor $\text{Sub} : \mathbf{V}(S) \to \mathbf{Poset}$ sending every $\mathcal{V}(S)$-algebra to its subalgebra lattice.

4. If $\mathbf{D}$ is a subcategory of $\mathbf{C}$, one can form the **injection functor** $I : \mathbf{D} \to \mathbf{C}$ sending every object and morphism in $\mathbf{D}$ to itself in $\mathbf{C}$. Thus $IA = A$ and $I(f) = f$ for $A \in \text{ob}(\mathbf{D}), f \in \text{hom}_{\mathbf{D}}(A, B)$. The special case when $\mathbf{D} = \mathbf{C}$ is the **identity functor** $1_{\mathbf{C}}$.

5. Fix an object $B \in \mathbf{D}$, then we have the **constant functor** $F : \mathbf{C} \to \mathbf{D}$ defined by $FA = B$ for all $A \in \text{ob}(\mathbf{C})$ and $F(f) = 1_B$ for all $f : A \to A'$ in $\mathbf{C}$.

6. If $\mathbf{C}$ is a discrete category, a functor $\mathbf{C} \to \mathbf{D}$ simply assigns each object of $\mathbf{C}$ an object of $\mathbf{D}$.

7. Define a functor $U : \mathbf{Ring} \to \mathbf{Grp}$ as follows: For each ring R, UR is its group of units. Since every ring homomorphism $f : R \to S$ sends units to units, it restricts to a group homomorphism $U(f) : UR \to US$.

8. For each ring R, one can form the polynomial ring $R[x]$ by adjoining a symbol x and agreeing that $xr = rx$ for all $r \in R$. Then every element of $R[x]$ is of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with the a_i's in R. Evidently a homomorphism $f : R \to S$ induces one $\overline{f} : R[x] \to S[x]$ by defining

$$\overline{f}(a_n x^n + \cdots + a_1 x + a_0) = f(a_n)x^n + \cdots + f(a_1)x + f(a_0)$$

This is a functor $\mathbf{Ring} \to \mathbf{Ring}$, sending every ring R to the polynomial ring $R[x]$ and every homomorphism f to $\overline{f}$.

9. Another functor $M_n : \mathbf{Ring} \to \mathbf{Ring}$ sends each ring R to the matrix ring $M_n(R)$ and each homomorphism $f : R \to S$ to the homomorphism $\tilde{f} : M_n(R) \to M_n(S)$ defined by

$$\tilde{f}(\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix}) = \begin{bmatrix} f(r_{11}) & f(r_{12}) & \dots & f(r_{1n}) \\ f(r_{21}) & f(r_{22}) & \dots & f(r_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ f(r_{n1}) & f(r_{n2}) & \dots & f(r_{nn}) \end{bmatrix}$$

10. Let $\mathbf{M}$ be a monoid M, regarded as a category with one object [Example 6 of Section 1]. Then a functor $\mathbf{M} \to \mathbf{Set}$ is an M-action. This is because $\mathbf{M}$ has only one object, so it is assigned by the functor to only one set. In fact, a functor $\mathbf{M} \to \mathbf{V}(S)$ is a $\mathcal{V}(S)$-representation of M.

11. Let R be a fixed ring, n a fixed positive integer. Whenever M is an R-module, M^n becomes a left module over the matrix ring $M_n(R)$ when defined as follows.

$$\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} r_{11}a_1 + r_{12}a_2 + \cdots + r_{1n}a_n \\ r_{21}a_1 + r_{22}a_2 + \cdots + r_{2n}a_n \\ \vdots \\ r_{n1}a_1 + r_{n2}a_2 + \cdots + r_{nn}a_n \end{bmatrix}$$

And for every R-module homomorphism $f : M \to N$, the map $f^n : M^n \to N^n$ sending $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \to \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_n) \end{bmatrix}$ is an $M_n(R)$-module homomorphism. Consequently, this defines a functor $R{-}\mathbf{mod} \to M_n(R){-}\mathbf{mod}$.

12. Example 1 shows that the takeoff $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ of varieties becomes a functor $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$. We proceed to construct a functor the other way $G : \mathbf{V}(S_2) \to \mathbf{V}(S_1)$. For each $A \in \mathbf{V}(S_2)$, there is a unique [up to isomorphism] universal $\mathcal{V}(S_1)$-algebra (U_A, i_A) enveloping B for the takeoff, by Theorems 1.28 and 1.29. Define $GA = U_A$. For $f : A \to B$ in G, to define $G(f) : U_A \to U_B$, note that $i_B f$ is an Ω_2-homomorphism $A \to U_B$. Therefore there exists a unique

Ω_1-homomorphism $h : U_A \to U_B$ such that $i_B f = h i_A$; define $G(f) = h$. Then $G(f)i_A = i_B f$ for all $f : A \to B$ in $\mathbf{V}(S_2)$, that is,

$$\begin{CD} A @>{f}>> B \\ @V{i_A}VV @VV{i_B}V \\ U_A @>{G(f)}>> U_B \end{CD}$$

is commutative.

Now suppose $f : A \to B$ and $g : B \to C$ in $\mathbf{V}(S_2)$. Then $G(gf)$ is the *unique* morphism $GA \to GC$ in $\mathbf{C}$ satisfying $G(gf)i_A = i_C gf$. However, $G(g)G(f)$ also satisfies this statement, because the commutativity of the squares in

$$\begin{CD} A @>{f}>> B @>{g}>> C \\ @V{i_A}VV @VV{i_B}V @VV{i_C}V \\ U_A @>{G(f)}>> U_B @>{G(g)}>> U_C \end{CD}$$

implies that

$$\begin{CD} A @>{gf}>> C \\ @V{i_A}VV @VV{i_C}V \\ U_A @>{G(g)G(f)}>> U_B \end{CD}$$

is commutative. Therefore, $G(gf) = G(g)G(f)$ by uniqueness. Likewise, for $A \in \mathcal{V}(S_2)$, $G(1_A)$ is the unique morphism $U_A \to U_A$ satisfying $G(1_A)i_A = i_A 1_A$. But obviously 1_{GA} satisfies that statement; whence $G(1_A) = 1_{GA}$. Therefore, G is a functor. It is a *left adjoint functor* of F, and that will be studied in Section 8.

The special case where $\mathcal{V}(S_2)$ is the variety of sets induces the **free-algebra functor** $G : \mathbf{Set} \to \mathbf{V}(S)$, sending every set X to $F_S(\Omega, X)$, and every set map $f : X \to Y$ the unique homomorphism $F_S(\Omega, X) \to F_S(\Omega, Y)$ which extends $i_Y f : X \to F_S(\Omega, Y)$.

13. A **functor in two variables** refers to a functor from a product category. For example, if $\mathbf{C}$ and $\mathbf{D}$ are categories, one cn form the **projection functor** $P_1 : \mathbf{C} \times \mathbf{D} \to \mathbf{C}$ by $P_1(A, B) = A$, $P_1(f, g) = f$. The other projection P_2 is defined similarly.

14. The **diagonal functor** $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ sends $A \to (A, A)$ and $f \to (f, f)$ for $f : A \to A'$ in $\mathbf{C}$.

A functor F is said to be **faithful** provided for all $A, B \in \mathrm{ob}(\mathbf{C})$, the map $\hom(A, B) \to \hom(FA, FB)$ sending $f \to F(f)$ is injective. In other words, $F(f) = F(g)$ for $f, g : A \to B$ imply $f = g$. But be careful; this *doesn't* mean F is an injective map on the objects. For example, the the functor in Example 1 is faithful, but not necessarily injective on the objects. Example 3 is both faithful and injective on the objects. Example 13 is not faithful though.

A functor F is said to be **full** provided that for all $A, B \in \mathrm{ob}(\mathbf{C})$, the map $\hom(A, B) \to \hom(FA, FB)$ sending $f \to F(f)$ is surjective. In other words, every morphism $FA \to FB$ is of the form $F(f)$ with $f : A \to B$. Take caution again: this doesn't mean F is a surjective map on the objects. The functor given by a takeoff [Example 1] is full if and only if the takeoff is full in the sense of Section 1.11, Exercise 8. Also, if $\mathbf{D}$ is a subcategory of $\mathbf{C}$, then the injection functor $\mathbf{D} \to \mathbf{C}$ is full if and only if $\mathbf{D}$ is a full subcategory, which explains the terminology.

If $\mathbf{C}$ and $\mathbf{D}$ are categories, a **contravariant functor** from $\mathbf{C}$ to $\mathbf{D}$ is a functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$. [$\mathbf{C}^{\mathrm{op}}$ is defined in Example 10 of Section 1.] Specifically, it's an object F such that:

(1) For each $A \in \mathrm{ob}(\mathbf{C})$, FA is some object of $\mathbf{D}$.

(2) For each $f : A \to B$ in $\mathbf{C}$, $F(f)$ is some morphism $FB \to FA$ of $\mathbf{D}$.

(3) $F(gf) = F(f)F(g)$ whenever gf is defined in $\mathbf{C}$.

(4) $F(1_A) = 1_{FA}$ for all $A \in \mathrm{ob}(\mathbf{C})$.

Statement (2), for example, is understood in the sense that $f \in \hom_{\mathbf{C}^{\mathrm{op}}}(B, A)$.

There are many kinds of functors which reverse the arrows like these, seen in the following examples. To avoid confusion, the plain word "functor" will always mean "covariant functor".

EXAMPLES

1. If all morphisms in $\mathbf{C}$ are isomorphisms, one can form a contravariant functor $\mathrm{Inv} : \mathbf{C} \to \mathbf{C}$ sending every object to itself and every morphism to its inverse. The conditions are readily verified. It is called the **inversion functor**.

2. If $\mathbf{M}$ and $\mathbf{N}$ are monoids viewed as one-object categories, a contravariant functor $\mathbf{M} \to \mathbf{N}$ is a monoid antihomomorphism. If $\mathbf{S}$ and $\mathbf{T}$ are preordered sets viewed as categories with at most one morphism in every hom set, a contravariant functor $\mathbf{S} \to \mathbf{T}$ is an order-reversing map.

3. Define a contravariant functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ as follows: For each $A \in \mathrm{ob}(\mathbf{Set})$, $\mathcal{P}A$ is the power set $\mathcal{P}(A)$, and for each set map $f : A \to B$, $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ sends every $X \subseteq B$ to its preimage $f^{-1}(X)$. Elementary set theory shows that $(gf)^{-1}(X) = f^{-1}(g^{-1}(X))$ when they are defined and $1_A^{-1}(X) = X$. Therefore, $\mathcal{P}$ is a contravariant functor.

In fact, since the preimage map is a Boolean algebra homomorphism [check this!] one could take the category of Boolean algebras as the codomain of $\mathcal{P}$, instead of $\mathbf{Set}$.

4. Let M be a fixed monoid. We define a contravariant functor $D : M{-}\mathbf{act} \to \mathbf{act}{-}M$ as follows. For each left M-action X, define $X^* = \hom(X, M)$, the set of M-action homomorphisms from X to M [with the obvious M-action structure]. To make X^* into a right M-action, take each $\varphi \in X^*, m \in M$. Define $\varphi m : X \to M$ by $\varphi m(x) = \varphi(x)m$ with $x \in X$. [This uses the monoid multiplication in M.] It is then straightforward that X^* is a right M-action. It is called the **dual** of the left M-action X.

Now suppose $f : X \to Y$ is a homomorphism of M-actions. Define the **transposed map** $f^* : Y^* \to X^*$ by $f^*(\varphi) = \varphi f$ with $\varphi \in Y^*$. We claim that

f^* is a homomorphism; for $\varphi \in Y^*, m \in M, x \in X$,

$$f^*(\varphi m)(x) = (\varphi m)(f(x)) = \varphi(f(x))m = f^*(\varphi)(x)m = (f^*(\varphi)m)(x)$$

Therefore, $f^*(\varphi m) = f^*(\varphi)m$. If one defines $DX = X^*, D(f) = f^*$, it is clear that D is a contravariant functor. It is called the **dual functor**.

One can likewise define a dual functor from $\mathbf{act}-M$ to $M-\mathbf{act}$. The details are left to the reader.

5. If R is a fixed ring, the same details from the previous example establish the dual functors $D : R-\mathbf{mod} \to \mathbf{mod}-R$ and $D : \mathbf{mod}-R \to R-\mathbf{mod}$: for every R-module M, let $M^* = \hom(M, R)$, and for every homomorphism $f : M \to N$, define $f^* : N^* \to M^*$ by $f^*(\varphi) = \varphi f$. But this time, one must take addition into account. Then assign $DM = M^*, D(f) = f^*$.

Do the categories form a category?

If $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ are functors, one defines the **composite functor** $GF : \mathbf{C} \to \mathbf{E}$ by $(GF)A = G(FA)$ for $A \in \mathrm{ob}(\mathbf{C})$, $(GF)(f) = G(F(f))$ for $f : A \to B$ in $\mathbf{C}$. Evidently this is a functor. Likewise, if F or G is contravariant, then GF can be defined this way: GF is covariant if F and G are both covariant or both contravariant; GF is contravariant if one of F, G is covariant and the other contravariant.

It is clear that $(HG)F = H(GF)$ when the compositions are defined, and when $F : \mathbf{C} \to \mathbf{D}$, $F1_{\mathbf{C}} = F = 1_{\mathbf{D}}F$. And certainly the domain and codomain of a functor are intrinsic. Doesn't this mean that *there's a category whose objects are categories and whose morphisms are covariant functors between categories*? They satisfy everything in the definition, don't they? But isn't it a bit fishy that one of the categories is to be made up of all of them? Well, the question is controversial. Some argue that the answer is "no":

(1) In the definition of a category, we said $\hom(A, B)$ has to be a set. However, the class of functors from $\mathbf{C}$ to $\mathbf{D}$ need not be a set.

(2) Just like a batch of sets doesn't necessarily form a set, a batch of classes — such as categories — might not be able to form a class. What they form would be a "conglomerate" at best.

(3) One would be able to form the full subcategory of categories that don't contain themselves as objects. This category contains itself as an object if and only if it doesn't, causing a paradox.

We shall be bold as to disagree with all those arguments [for example, (3) doesn't work because set-builder notation can't be used on a proper class]. Thus we assume that the category of all categories can be formed, and we call it **Cat**. Meow if you find this cute!

Natural Transformations

If you're overburdened by the fact that categories [whose role is to hold morphisms] have morphisms of their own [the functors], prepare to know that functors have their own morphisms as well! These morphisms, called natural transformations, actually have the slightest bit of information in their hands. If $F, G : \mathbf{C} \to \mathbf{D}$ are functors, a natural transformation from F to G assigns each object A in $\mathbf{C}$ a morphism $FA \to GA$ such that the morphisms have the "same state of mind" and are therefore compatible with where F and G send morphisms. This is formalized in the definition:

DEFINITION

Let $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $F, G : \mathbf{C} \to \mathbf{D}$ *functors. A* ***natural transformation*** *from* F *to* G *is a mathematical object* η *assigning each* $A \in ob(\mathbf{C})$ *to a morphism* $\eta_A \in \hom_{\mathbf{D}}(FA, GA)$ *such that for all* $f : A \to B$ *in* $\mathbf{C}$*, the diagram*

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

is commutative. If every η_A *is an isomorphism,* η *is called a* ***natural isomorphism****.*

EXAMPLES

1. Let $T : \mathbf{Ring} \to \mathbf{Mon}$ be the takeoff from rings to monoids, regarding the multiplication and forgetting the addition. Then, let $U : \mathbf{Ring} \to \mathbf{Mon}$ be the functor of Example 6 — since groups are monoids, we can change the codomain this way! For each ring R, TR is the multiplicative monoid of R, whereas UR is the group of units of R. It turns out that UR is solely the group of units of the monoid TR, and one can consider the canonical monomorphism $\eta_R : UR \hookrightarrow TR$ in **Mon**. It is straightforward to see that if η_R is defined that way for every ring R, η is a natural transformation from U to T.

2. Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties, and $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$, $G : \mathbf{V}(S_2) \to \mathbf{V}(S_1)$ be the functors in Examples 1 and 12. For each $A \in \mathcal{V}(S_2)$, GA is the universal $\mathcal{V}(S_1)$ algebra enveloping A, let $i_A : A \to GA$ be the $\mathcal{V}(S_2)$ map. Then i_A is really a morphism from A to FGA because it regards the $\mathcal{V}(S_1)$ algebra GA with the derived structure. The statement $G(f)i_A = i_B f$, which is really $FG(f)i_A = i_B f$, says that $A \to i_A$ is a natural transformation from $1_{\mathbf{V}(S_2)}$ to FG.

3. Fix a monoid M, and recall the dual functors $D : M{-}\mathbf{act} \to \mathbf{act}{-}M$ and $\mathbf{act}{-}M \to M{-}\mathbf{act}$. Composing them yields the **double dual functor** $D^2 : M{-}\mathbf{act} \to M{-}\mathbf{act}$. It sends each M-action X to $X^{**} = \hom(\hom(X, M), M)$. For each X, define $\eta_X : X \to X^{**}$ by agreeing that $\eta_X(x)$ for each $x \in X$ is the map $\hom(X, M) \to M$ sending $\varphi \to \varphi(x)$. Thus $\eta_X(x)(\varphi) = \varphi(x)$. We claim that η is a natural transformation from $1_{M-\mathbf{act}} \Rightarrow D^2$.

First we must show that η_X has a good target in the sense that $\eta_X(x)$ is actually a homomorphism of right M-actions. To do this, we need to show that $\eta_X(x)(\varphi m) = \eta_X(x)(\varphi)m$. Well, $\eta_X(x)(\varphi m) = \varphi m(x) = \varphi(x)m$ (by definition of $\varphi m) = \eta_X(x)(\varphi)m$.

Next we show that each η_X is a homomorphism of left M-actions: For all $x \in X, m \in M, \varphi \in X^*$, $\eta_X(mx)(\varphi) = \varphi(mx) = m\varphi(x) = m\eta_X(x)(\varphi)$, since $\varphi \in \hom(X, M)$ is a left M-action homomorphism. Therefore, $\eta_X(mx) = m\eta_X(x)$.

Finally we claim that η is a natural transformation from $1_{M-\mathbf{act}}$ to D^2. Let $f : X \to Y$ be any M-action homomorphism. Then $f^* : Y^* \to X^*$ is given by $f^*(\varphi) = \varphi f$, and $f^{**} : X^{**} \to Y^{**}$ is given by $f^{**}(\psi) = \psi f^*$. Thus $f^{**}(\psi)(\varphi) = \psi f^*(\varphi) = \psi(f^*(\varphi)) = \psi(\varphi f)$. To show that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ Y & \xrightarrow{\eta_Y} & Y^{**} \end{array}$$

is commutative, we work as follows:

$$f^{**}(\eta_X(x))(\varphi) = \eta_X(x)(\varphi f) = \varphi f(x) = \varphi(f(x))$$

$$\eta_Y f(x)(\varphi) = \eta_Y(f(x))(\varphi) = \varphi(f(x))$$

Therefore, $f^{**}(\eta_X(x))(\varphi) = \eta_Y(f(x))(\varphi)$ for all $x \in X, \varphi \in Y^*$. Hence $f^{**}(\eta_X(x)) = \eta_Y(f(x))$ for all x, and $f^{**}\eta_X = \eta_Y f$. Therefore, η is a natural transformation.

The foregoing can be done with modules as well as monoid actions.

If $F, G : \mathbf{C} \to \mathbf{D}$ are contravariant functors, one can still define a natural transformation from F to G. This time, it's an assignment $A \to \eta_A$ with $\eta_A \in \hom_{\mathbf{D}}(FA, GA)$, such that for all $f : A \to B$ in $\mathbf{C}$,

$$\begin{array}{ccc} FB & \xrightarrow{\eta_B} & GB \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FA & \xrightarrow{\eta_A} & GA \end{array}$$

is commutative. This need not be studied seperately, though, because F and G are actually covariant functors from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$.

Let $F, G, H : \mathbf{C} \to \mathbf{D}$ be functors, $\eta : F \Rightarrow G$ and $\zeta : G \Rightarrow H$ natural transformations. Then it is clear that the assignment $A \to \zeta_A\eta_A$ is a natural transformation from F to H. It is notated as $\zeta\eta$ and is called the **composite natural transformation**. Also, we have the **identity natural transformation** 1_F from F to F assigning $A \to 1_{FA}$. Evidently $(\theta\zeta)\eta = \theta(\zeta\eta)$ and $1_G\eta = \eta = \eta 1_F$ when they are defined. It therefore follows that the covariant functors from $\mathbf{C}$ to $\mathbf{D}$ form a category, whose morphisms are natural transformations of functors. [We temporarily allow $\hom(F, G)$ to be a proper class here.] It is called the **functor category** of $\mathbf{C}$ to $\mathbf{D}$ and is notated $\mathbf{D}^{\mathbf{C}}$.

Evidently in $\mathbf{D}^{\mathbf{C}}$, isomorphisms are natural isomorphisms of functors.

EXERCISES

1. Let $F : \mathbf{C} \to \mathbf{D}$ be a covariant functor.

   (a) F preserves isomorphisms; that is, if $f : A \to B$ is an isomorphism in $\mathbf{C}$, then $F(f) : FA \to FB$ is an isomorphism in $\mathbf{D}$.

   (b) F also preserves sections and retractions.

   (c) If $F(f)$ is monic and F is faithful, then f is monic. Give a counterexample when F is not faithful.

   (d) If f is monic and F is full, then $F(f)$ is monic. Give a counterexample when F is not full.

   (e) Repeat parts (c) and (d) with "monic" replaced with "epic".

   (f) What if F is contravariant? Modify parts (a)-(e) to hold for contravariant F.

2. If $\mathbf{Q}$ is any class of objects which assigns any two objects A, B a set $\hom(A, B)$, but there's no notion of composition of morphisms or identity morphisms, $\mathbf{Q}$ is called a **quiver**. Thus a category is a quiver when that information is disregarded in it.

   If $\mathbf{Q}$ and $\mathbf{R}$ are quivers, a map from $\mathbf{Q}$ to $\mathbf{R}$ is kind of like a functor without the structure to preserve: it assigns each $A \in \mathrm{ob}(\mathbf{Q})$ an object FA in $\mathbf{R}$ and each $f : A \to B$ in $\mathbf{Q}$ a map $F(f) : FA \to FB$ in $\mathbf{R}$.

   (a) For each quiver $\mathbf{Q}$, the define the **free category C** given by $\mathbf{Q}$ as follows: $\mathrm{ob}(\mathbf{C}) = \mathrm{ob}(\mathbf{Q})$, and for any objects A, B, $\hom_{\mathbf{C}}(A, B)$ is the set of all strings of the form $f_n \dots f_2 f_1$ with $f_1 : A \to A_1, f_2 : A_1 \to A_2, \dots f_n : A_{n-1} \to B$ in $\mathbf{Q}$. When $A = B$, the empty string is included as 1_A. Define the composition of morphisms by the obvious juxtaposition. Verify that $\mathbf{C}$ is indeed a category.

   (b) Let $I : \mathbf{Q} \to \mathbf{C}$ be the map sending objects to themselves and morphisms to themselves as one-element strings. If $\mathbf{D}$ is a category and $J : \mathbf{Q} \to \mathbf{D}$ is a map, there is a unique functor $F : \mathbf{C} \to \mathbf{D}$ such that $J = FI$.

3. Let U be the functor **Ring** $\to$ **Grp** of Example 6, and M_n the functor from **Ring** $\to$ **Ring** of Example 8. Let GL_n be the composite functor UM_n; to what does it send a ring R?

4. Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties, and F, G the functors given in Examples 1 and 12.

   (a) For each $A \in \mathcal{V}(S_1)$, regard A as a $\mathcal{V}(S_2)$-algebra with the derived structure, and let (U, i) be a universal enveloping A for the takeoff T. Show that $i : A \to U$ has a unique retraction $r_A : U \to A$ which is an Ω_1-homomorphism.

   (b) Explain why $r_A \in \hom_{\mathbf{V}(S_1)}(GFA, A)$.

   (c) Show that $A \to r_A$ is a natural transformation from GF to $1_{\mathbf{V}(S_1)}$.

5. (a) Give two examples of covariant functors $\mathbf{V}(S) \to \mathbf{V}(S)-\mathbf{sub}$. [$\mathbf{V}(S)-\mathbf{sub}$ is defined in Exercise 7 of Section 1.]

   (b) Give two examples of covariant functors $\mathbf{V}(S)-\mathbf{sub} \to \mathbf{V}(S)$.

   (c) Do parts (a) and (b) for $\mathbf{V}(S)-\mathbf{con}$.

   (d) Now express canonical monomorphisms and canonical epimorphisms in the form of natural transformations.

6. Let $\mathbf{C}$ be a category, A an object of $\mathbf{C}$. Let $\mathbf{C}/A$ be the category in Example 11 of Section 1. Define $F : \mathbf{C}/A \to \mathbf{C}$ sending $(B, f) \to B$ and each morphism u to itself. Then F is a functor.

7. Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors, and suppose for each $A \in \mathrm{ob}(\mathbf{C})$, $\eta_A : FA \to GA$ is any morphism. [This is called an **infranatural transformation**.] Show that $\mathbf{C}_\eta$ is a subcategory of $\mathbf{C}$ when defined by $\mathrm{ob}(\mathbf{C}_\eta) = \mathrm{ob}(\mathbf{C})$, and each $f \in \hom_{\mathbf{C}}(A, B)$ is in $\hom_{\mathbf{C}_\eta}(A, B)$ if and only if

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

   is commutative. [$\mathbf{C}_\eta$ is referred to as the **naturalizer** of η.]

8. Let $F, G : \mathbf{C} \to \mathbf{D}, H : \mathbf{D} \to \mathbf{E}, K : \mathbf{B} \to \mathbf{C}$ be functors. Let $\eta : F \Rightarrow G$ be a natural transformation.

   (a) $H\eta : HF \Rightarrow HG$ is a natural transformation given by $A \to H(\eta_A)$ for $A \in \mathrm{ob}(\mathbf{C})$.

   (b) $\eta K : FK \Rightarrow GK$ is a natural transformation given by $A \to \eta_{KA}$ for $A \in \mathrm{ob}(\mathbf{B})$.

   Now assume capital letters are functors and lowercase Greek letters are natural transformations.

   (c) The products are *functorial*:

$$F(\zeta\eta) = (F\zeta)(F\eta), F1_G = 1_{FG}, (\zeta\eta)G = (\zeta G)(\eta G), 1_F G = 1_{FG}$$

   when they are defined. [*Hint*: Just use the definition!]

   (d) The products form a *biaction*:

$$(GF)\eta = G(F\eta), 1_{\mathbf{D}}\eta = \eta, \eta(GF) = (\eta G)F, \eta 1_{\mathbf{C}} = \eta, (G\eta)F = G(\eta F)$$

   when they are defined.

   (e) Suppose $F, F' : \mathbf{C} \to \mathbf{D}$, $G, G' : \mathbf{D} \to \mathbf{E}$ are functors and $\eta : F \Rightarrow F'$, $\zeta : G \Rightarrow G'$ are natural transformations. Show that $(\zeta F')(G\eta) = (G'\eta)(\zeta F)$ as natural transformations $GF \Rightarrow G'F'$. [*Hint*: This comes from the naturality of one of them.]

9. Express Example 9 of Section 1 in the form of a functor category.

10. If $\mathbf{C}$ and $\mathbf{D}$ are categories and $A \in \text{ob}(\mathbf{C})$ is fixed, then $P : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$ given by $F \to FA, \eta \to \eta_A$ is a functor. It is called the **projection onto** A **of the functor category**.

11. Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be categories, $F : \mathbf{D} \to \mathbf{E}$ a functor. Use Exercise 8 to define functors $F^{\mathbf{C}} : \mathbf{D}^{\mathbf{C}} \to \mathbf{E}^{\mathbf{C}}$ and $\mathbf{C}^F : \mathbf{C}^{\mathbf{E}} \to \mathbf{C}^{\mathbf{D}}$.

12. (a) A functor $F : \mathbf{C} \to \mathbf{D}$ is an **isomorphism** provided there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $GF = 1_{\mathbf{C}}$ and $FG = 1_{\mathbf{D}}$. Informally, what can you say about isomorphic categories?

    (b) Let $\cong$ denote natural isomorphism of functors here. If $F \cong G$, then $HF \cong HG$ and $FK \cong GK$ when they are defined. [*Hint*: Exercise 11 gives a shortcut.]

    (c) A functor $F : \mathbf{C} \to \mathbf{D}$ is an **equivalence** if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$. $\mathbf{C}$ and $\mathbf{D}$ are **equivalent** if such a functor F exists. Show that this is an equivalence relation on the categories. Also, isomorphic categories are equivalent, but not conversely.

    (d) A functor $F : \mathbf{C} \to \mathbf{D}$ is an equivalence if and only if F is faithful and full and for every $B \in \text{ob}(\mathbf{D})$ there exists $A \in \mathbf{ob}(\mathbf{C})$ such that FA and B are isomorphic in $\mathbf{D}$. [*Hint*: $\Rightarrow$ If $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$, then FG and GF are faithful and full [why?] Use this to prove that F and G are faithful and full. Also show that for $B \in \text{ob}(\mathbf{D})$, FGB is isomorphic to B. $\Leftarrow$ Define $G : \mathbf{D} \to \mathbf{C}$ sending each B to some A such that there exists an isomorphism $\sigma_B : FA \to B$. Show that there is a unique way for G to assign morphisms so that σ is a natural isomorphism $FG \cong 1_{\mathbf{D}}$. To show $GF \cong 1_{\mathbf{C}}$, note that for each $A \in \text{ob}(\mathbf{C})$ there is a unique isomorphism $\eta_A : GFA \to A$ such that $F(\eta_A) = \sigma_{FA}$.]

13. Recall that if A, B, C are $\mathcal{V}(S)$ algebras, $(A \times B) \times C \cong A \times (B \times C)$ due to an isomorphism $\sigma_{A,B,C}$ sending $((a, b), c) \to (a, (b, c))$. Show that this isomorphism is *natural* in the sense that for all $f : A \to A', g : B \to B', h : C \to C'$ this diagram is commutative:

$$\begin{array}{ccc} (A \times B) \times C & \xrightarrow{\sigma_{A,B,C}} & A \times (B \times C) \\ \Big\downarrow{\scriptstyle (f \times g) \times h} & & \Big\downarrow{\scriptstyle f \times (g \times h)} \\ (A' \times B') \times C' & \xrightarrow{\sigma_{A',B',C'}} & A' \times (B' \times C') \end{array}$$

    Here $f \times g$ denotes the product of maps; that is, $(f \times g)(a, b) = (f(a), g(b))$.

14. Define the **center** of a category $\mathbf{C}$ to be the class of natural transformations from $1_{\mathbf{C}}$ to $1_{\mathbf{C}}$; that is, $\text{hom}(1_{\mathbf{C}}, 1_{\mathbf{C}})$ in the functor category $\mathbf{C}^{\mathbf{C}}$. Evidently the center has a monoidal structure under composition of natural transformations.

Now let M be a fixed monoid, $C(M) = \{a \in M \mid ax = xa \ \forall x \in M\}$ be the center of M. For $c \in C(M)$, the assignment of each M-action X to the homomorphism $x \to cx$ from $X \to X$ is a natural transformation $\eta(c)$ from $1_{M-\mathbf{act}}$ to itself. Furthermore, the map $c \to \eta(c)$ is a monoid isomorphism from $C(M)$ into the center of $M-\mathbf{act}$.