2.2 - Monic and Epic Morphisms

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(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

The injectivity and surjectivity of $\mathcal{V}(S)$-algebras was quite important. This importance carries over to category theory. Unfortunately, for morphisms which don't go from sets to sets, injectivity and surjectivity can hardly be defined. Let's explore the $\mathcal{V}(S)$-algebras, once again.

Suppose $f : A \to B$ is an injective homomorphism in $\mathbf{V}(S)$. If $C \in \mathcal{V}(S)$, $g, h : C \to A$ and $fg = fh$, then for all $a \in A$, $fg(a) = fh(a)$. Thus $f(g(a)) = f(h(a))$, which implies $g(a) = h(a)$ since f is injective. Therefore, $g = h$.

But what if f isn't injective? Let $C = \ker f = \{(a, b) \in A \times A \mid f(a) = f(b)\}$. Then $C \in \mathcal{V}(S)$. Define $g, h : C \to A$ by $g(a, b) = a$ and $h(a, b) = b$. For all $(a, b) \in C$, $f(a) = f(b)$ by definition, so that $fg(a, b) = fh(a, b)$. Therefore, $fg = fh$. However, since f is not injective, there exist $a \neq b$ in A such that $f(a) = f(b)$. Furthermore, $(a, b) \in C$ and $g(a, b) \neq h(a, b)$, so that $g \neq h$.

We have shown

A homomorphism $f : A \to B$ in $\mathbf{V}(S)$ is injective if and only if $fg = fh$ always implies $g = h$ for homomorphisms $g, h : C \to A$.

This defines injectivity of a homomorphism using purely maps, leading to the following definition and theorem.

DEFINITION

*If $\mathbf{C}$ is a category, and $f : A \to B$ in $\mathbf{C}$, f is said to be **monic** provided that whenever $fg = fh$ in $\mathbf{C}$, $g = h$.*

THEOREM 2.1a *A homomorphism in $\mathbf{V}(S)$ is monic if and only if it's injective.*

How how would one define surjectivity? There is a bit of less luck here. Suppose $f : A \to B$ is surjective in $\mathbf{V}(S)$. If $C \in \mathcal{V}(S)$, $g, h : B \to C$ and $gf = hf$, then for all $b \in B$, $b = f(a)$ for some $a \in A$ since f is surjective, and $g(b) = gf(a) = hf(a) = h(b)$. Hence, $g = h$.

But in rare cases, we could still have $gf = hf \implies g = h$ when f is not surjective. Consider the canonical monomorphism of rings $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$. If $g : \mathbb{Q} \to R$ and $h : \mathbb{Q} \to R$ are ring homomorphisms such that $gf = hf$, then $g|\mathbb{Z} = h|\mathbb{Z}$. Exercise 1 shows that $g = h$ follows. So this property of maps *strictly contains* surjectivity:

DEFINITION

*If $\mathbf{C}$ is a category, and $f : A \to B$ in $\mathbf{C}$, f is said to be **epic** provided that whenever $gf = hf$ in $\mathbf{C}$, $g = h$.*

The preceding example shows that the canonical monomorphism $\iota : \mathbb{Z} \to \mathbb{Q}$ in **Ring** is epic but not surjective, so we have:

THEOREM 2.1b *A surjective homomorphism in* $\mathbf{V}(S)$ *is epic, but not conversely.*

A complete classification of epic homomorphisms in $\mathbf{V}(S)$ is found in Exercise 2.

Theorem 2.1a may be false when the category consists of Ω-algebras but isn't a variety. To see an example, define a **divisible abelian group** to be an abelian group G [written additively] such that for all $a \in G$, $0 \neq n \in \mathbb{Z}$, there exists $b \in G$ with $a = nb$. Thus you can divide by any nonzero integer in G.

Let $\mathbf{Ab} - \mathbf{div}$ be the full subcategory of $\mathbf{Ab}$ consisting of the divisible abelian groups. Clearly $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible; let $f : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the canonical epimorphism. We claim that f is monic, even though it is not injective: suppose G is a divisible abelian group and $g, h : G \to \mathbb{Q}$ are homomorphisms with $fg = fh$. Then $g - h : G \to \mathbb{Q}$ is a homomorphism and $f(g - h) = 0$, implying that $\operatorname{im}(g - h) \subseteq \ker f = \mathbb{Z}$. Evidently a homomorphic image of a divisible abelian group is divisible, so $\operatorname{im}(g - h)$ is divisible [since G is]. The only subgroup of $\mathbb{Z}$ which is divisible is 0, and hence, $g - h$ is the zero map and $g = h$. Therefore, f is monic.

The conclusion is that if categories consist of sets and maps, monicness and epicness are only *approximations* of injectivity and surjectivity.

Now for a few basic properties about morphisms in an arbitrary category. For example, we know that if $f : A \to B$ and $g : B \to C$ are both injective in **Set**, gf is injective. The same is true for monic morphisms, as we now see.

THEOREM 2.2 *Let* $\mathbf{C}$ *be a category and* $f : A \to B$, $g : B \to C$ *morphisms in* $\mathbf{C}$.

(1) *If* f *and* g *are monic,* gf *is monic.*
(2) *If* gf *is monic, then* f *is monic.*
(3) *If* f *and* g *are epic,* gf *is epic.*
(4) *If* gf *is epic, then* g *is epic.*

Proof of Theorem 2.2. (1) If $gfx = gfy$ with $x, y : D \to A$, then $fx = fy$ since g is monic, hence $x = y$ since f is monic.

(2) If $fx = fy$ with $y : D \to A$, then $gfx = gfy$, so that $x = y$ since gf is monic.

(3) and (4) have essentially the same proof with the arrows reversed. ■

We now define subobjects and quotient objects of an arbitrary object in a category.

Fix $A \in \operatorname{ob}(\mathbf{C})$. Consider the class of all monic morphisms from any object in $\mathbf{C}$ to A. If f and g are such morphisms, define $f \leq g$ provided that $f = gk$ for some k [which is theoretically monic]. Then evidentiy $\leq$ is reflexive and transitive. Now define $f \equiv g$ provided that $f \leq g$ and $g \leq f$; equivalently,

$f = gk$ for some *isomorphism* k. $\equiv$ is an equivalence relation, and its equivalence classes are the **subobjects** of A.

Furthermore, if $f \equiv f'$ and $g \equiv g'$, then $f \leq g$ if and only if $f' \leq g'$. Thus $\leq$ is actually a partial order on the subobjects of A.

The special case where $\mathbf{C} = \mathbf{V}(S)$ considers all injective homomorphisms into A. If f and g are such homomorphisms, $f \leq g$ if and only if $\text{im } f \subseteq \text{im } g$. Thus, $f \equiv g$ if and only if they have the same image. Since the possible images of injective homomorphisms into A are the subalgebras of A, the above definition makes sense.

Now consider the class of all epic morphisms from A to any object in $\mathbf{C}$. If f and g are such morphisms, define $f \geq g$ to mean $f = kg$ for some k, and $f \equiv g$ to mean $f \geq g$ and $g \geq f$. Once again, $\equiv$ is an equivalence relation, its equivalence classes are the **quotient objects** of A, and $\geq$ is a partial order on them.

If f and g are surjective homomorphisms from A in $\mathbf{V}(S)$, then $f \geq g$ if and only if $\ker f \supseteq \ker g$, and $f \equiv g$ if and only if they have the same kernel. Since the possible kernels are the congruence relations on A, the above definition would make sense, except for the trap that epic homomorphisms need not be surjective. Because of this, the class of quotient objects of A as defined above may be larger than the actual class of quotient algebras.

CAUTION In universal algebra, people refer to injective homomorphisms as **monomorphisms**, and surjective homomorphisms as **epimorphisms**. In category theory, however, a "monomorphism" refers to a monic morphism and an "epimorphism" refers to an epic morphism. You need to watch out which definitions are being used when, because they are not the same!

EXERCISES

1. Suppose $g : \mathbb{Q} \to R$ and $h : \mathbb{Q} \to R$ are ring homomorphisms with $g|\mathbb{Z} = h|\mathbb{Z}$.

   (a) For each $n \neq 0$ in $\mathbb{Z}$, $g(n)$ is a two-sided unit in R. [*Hint*: Multiply by $g(1/n)$ on both sides.]

   (b) For each $n \neq 0$ in $\mathbb{Z}$, $g(1/n) = h(1/n)$.

   (c) $g = h$. Hence, the canonical monomorphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$, though not surjective, is epic.

2. (UNIVERSAL ALGEBRA) If A and B are $\mathcal{V}(S)$-algebras and $g, h : A \to B$ are homomorphisms, let $K = \{a \in A \mid g(a) = h(a)\}$. We already know that K is a subalgebra of A [Exercise 10(a) of Section 1.3]. K is called the **difference kernel** of g and h.

   (a) If $f : C \to A$ is a homomorphism, then $gf = hf$ if and only if $f(C) \subseteq K$.

   (b) If B is a subalgebra of A, then B is said to be **nice** provided that there exists an $\mathcal{V}(S)$-algebra C and homomorphisms $g, h : A \to C$ with difference kernel B. Show that A is nice in A, and the intersection of any batch of nice subalgebras of A is nice. [*Hint*: Products.] Conclude that Theorem 1.3 can be applied to define the nice subalgebra generated by a set.

(c) A homomorphism $f : A \to B$ is epic if and only if B is the *nice* subalgebra of B generated by $f(A)$.

(d) Every submodule of an R-module is nice. Conclude that epic morphisms are surjective in $R-$**mod**.

(e) If R is a ring and S is a nice subring of R, then whenever $u \in S$ is a unit in R, then $u^{-1} \in S$. Conclude that $\mathbb{Z}$ is *not* a nice subring of $\mathbb{Q}$.

3. (UNIVERSAL ALGEBRA) If A and B are $\mathcal{V}(S)$-algebras and $g, h : A \to B$ are homomorphisms, let Θ be the congruence relation on B generated by $\{(f(a), g(a)) \mid a \in A\}$. Θ is called the **difference image** of f and g.

   (a) If $f : B \to C$ is a homomorphism, then $fg = fh$ if and only if $\Theta \subseteq \ker f$.

   (b) For *every* congruence relation Φ on B, there exists an $\mathcal{V}(S)$-algebra A and homomorphisms $g, h : A \to B$ with difference image Φ. Explain why this implies that monics are injective in $\mathbf{V}(S)$.

4. In **Grp**, epic morphisms are surjective. [*Hint*: Suppose $f : G \to H$ is a group homomorphism which is not surjective. If $I = f(G)$ is normal in H, the proof that f is not epic should be easy. Otherwise, $[H : I] \geq 3$ [why?] Let $S(H)$ be the group of permutations of the *set* H and define $g : H \to S(H)$ by sending every $a \in H$ to $a_L \in S(H)$ sending $x \to ax$. Show that there exists $p \in S(H)$ which commutes with every a_L with $a \in I$, but fails to commute with some a_L with $a \in H - I$. Then, if $h : H \to S(H)$ is defined by $h(a) = pa_Lp^{-1}$, show that $gf = hf$ but $g \neq h$.]

5. If $f : A \to B$ and $g : B \to A$ are morphisms in $\mathbf{C}$ such that $gf = 1_A$, f is said to be a **section** of g and g is said to be a **retraction** of f. Hence, a morphism is a section if and only if it has a retraction, and a morphism is a retraction if and only if it has a section.

   (a) Sections are monic and retractions are epic in $\mathbf{C}$, but not conversely.

   (b) Show by example that a morphism in $\mathbf{C}$ may have more than one section, or more than one retraction.

   (c) If f has both a section and a retraction, then f is an isomorphism, and the conditions in (b) can't hold.

   (d) Comment on how this links to universal algebra. [*Hint*: See Exercise 10 of Section 1.5]