2.1 - Definition and Examples of Categories

Nicholas McConnell

(Categories)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

The first chapter generalized the notion of an algebraic structure, and dealt with homomorphisms in between them. Now we do something even weirder: we generalize the notion of a homomorphism, not necessarily between sets!

A category consists of a community of objects and morphisms between them. It does not have any trivial structure we have previously dealt with. However, given a category, we can define many things reasonably, and give proofs which focus on morphisms. It just comes to a challenge that we cannot treat the objects like sets [until Section 9 comes along].

This is an ubiquitous concept in mathematics. It starts out involving algebraic structures, but then changes to entirely different structures, like the ones in the later chapters. What's more awkward is that categories have morphisms of their own [Section 3], and these morphisms have *their* own morphisms!

To see what morphisms would look like, recall the basic properties of homomorphisms of $\mathcal{V}(S)$ algebras.

To begin with, the domain and codomain of a homomorphism are intrinsic, even though surjectification is possible. [For example, the set map $\{0\} \to \mathbb{Z}$ sending 0 to 2 is different from the map $\{0\} \to \mathbb{R}$ sending 0 to 2.] This is a rule which prevents any confusion in category theory.

The second thing to realize is that if $f : A \to B$ and $g : B \to C$ are homomorphisms, so is $gf : A \to C$ [Theorem 1.4(1)]. Reread the statement in the proof if you don't remember why.

Next, if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(hg)f = h(gf)$, because both send $x \in A$ to $h(g(f(x))) \in D$. It is well-known that composition of functions is associative, no matter what the functions are.

The final thing about homomorphisms is that $1_A : A \to A$ is a homomorphism [Theorem 1.4(2)], and whenever $f : A \to B$ and $g : B \to A$, clearly $f1_A = f$ and $1_A g = g$.

Abstracting the properties just gone over:

DEFINITION

*A **category** is a mathematical object* $\mathbf{C}$ *with new structure given by the following:*

(1) *ob*($\mathbf{C}$) *is a class, whose elements are called the **objects** of* $\mathbf{C}$.

(2) *For each* $A, B \in ob(\mathbf{C})$, $\hom_{\mathbf{C}}(A, B)$ *[or* $\hom(A, B)$ *if* $\mathbf{C}$ *is clearly under discussion] is a set whose elements are called **morphisms** from* A *to* B*. One writes* $f : A \to B$ *for* $f \in \hom(A, B)$.

(3) *If* $(A, B) \neq (A', B')$, $\hom(A, B)$ *and* $\hom(A', B')$ *are disjoint.*

(4) *Whenever* $f : A \to B$ *and* $g : B \to C$*, the **composite function*** gf *is some morphism* $A \to C$*. Stated otherwise, for each* $A, B, C \in ob(\mathbf{C})$*, a map* $\hom(B, C) \times \hom(A, B) \to \hom(A, C)$ *is equipped.*

(5) [associativity] *When* $f : A \to B$, $g : B \to C$ *and* $h : C \to D$, $(hg)f = h(gf)$. *As usual, we simplify this to* hgf.

(6) [identity] *For each* $A \in ob(\mathbf{C})$, *there is a morphism* $1_A : A \to A$ *such that whenever* $f : A \to B$ *and* $g : B \to A$, $f1_A = f$ *and* $1_A g = g$. 1_A *is called the* ***identity morphism*** *on* A.

REMARKS Condition (3) is useful, but it is not very necessary. Whatever sets the $\hom(A, B)$'s are, their elements could be tagged indicating where they are, making the sets disjoint. And the identity morphism 1_A is unique, because if $1'_A : A \to A$ also satisfies the condition, $1_A = 1_A 1'_A = 1'_A$.

Since this definition is hard to understand, examples would surely help.

EXAMPLES

1. A variety $\mathcal{V}(S)$ becomes a category $\mathbf{V}(S)$ with $\mathrm{ob}(\mathbf{V}(S))$ the class of $\mathcal{V}(S)$ algebras, and $\hom(A, B)$ the set of homomorphisms $A \to B$, where composition of morphisms is the usual function composition and $1_A : A \to A$ is the identity map on A. [Note that $\hom(A, B)$ may be empty.] In particular, varieties we know already yield the categories **Mon** [monoids], **Grp** [groups], **Ab** [abelian groups], **Ring** [rings], **Rng** [rngs], **Rinv** [rings with involution], $R-$**mod** [left R-modules with R a given ring], $M-$**act** [left M-actions], and lots more ... and, of course, **Set**, the category of sets.

2. In fact, using Exercise 5 of Section 1.11, one can form the category of all varieties **Var** where the morphisms are takeoffs. This category is quite complicated, because of the possible difficulty in verifying the axioms.

3. The integral domains form a category **Dom** where homomorphisms are the usual ring homomorphisms. Note, however, that the integral domains *don't* form a variety.

4. The fields form a category **Field** where homomorphisms are the usual ring homomorphisms. Note, by the way, that all of the homomorphisms are injective! [Exercise 1] Similarly, if F is a particular field, one could form the category F**-Ext** of extension fields of F, where only homomorphisms that send every element of F to itself are admitted.

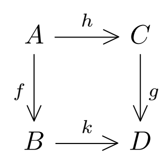
5. A category $\mathbf{C}$ is said to be **discrete** provided $\hom(A, B) = \emptyset$ when $A \neq B$ and $\hom(A, A) = \{1_A\}$. Discrete categories can be identified purely with their objects.

6. Notice that if A is an object in $\mathbf{C}$, $\hom(A, A)$ is a monoid with the categorical structure. Every monoid can be found this way: let M be a monoid. Define $\mathbf{M}$ by saying that $\mathrm{ob}(\mathbf{M}) = \{A\}$, and $\hom(A, A) = M$, where composition of morphisms agree with the binary operation in M and the identity morphism is $1 \in M$. Then the validity of the axioms is clear.

7. Let S be a set with a preorder [i.e. reflexive and transitive] relation $\leq$. Define a category $\mathbf{S}$ by $\mathrm{ob}(\mathbf{S}) = S$, and when $a, b \in S$, $\hom(a, b)$ has exactly one morphism if $a \leq b$, otherwise $\hom(a, b) = \emptyset$. Then $\mathbf{S}$ is a category with composition and identity maps unique determined, and is said to be a **category given by a preorder**.

8. [New categories from old] Let $\mathbf{C}$ and $\mathbf{D}$ be arbitrary categories. Define a new category $\mathbf{C} \times \mathbf{D}$ by specifying that $\mathrm{ob}(\mathbf{C} \times \mathbf{D})$ is the class of pairs (A, B) with $A \in \mathrm{ob}(\mathbf{C})$ and $B \in \mathrm{ob}(\mathbf{D})$, and $\hom_{\mathbf{C}\times\mathbf{D}}((A, B), (C, D)) = \hom_{\mathbf{C}}(A, C) \times \hom_{\mathbf{D}}(B, D)$, the set of pairs of the form (f, g) where $f : A \to C$ and $g : B \to D$. Define $(g, g_1)(f, f_1) = (gf, g_1 f_1)$ when possible and $1_{(A,B)} = (1_A, 1_B)$. This is easily seen to be a category. It is called the **product category of C and D**.

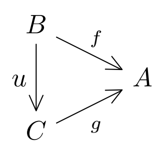
9. If $\mathbf{C}$ is an arbitrary category, the objects in $\mathbf{C}^{\to}$ are the morphisms in $\mathbf{C}$, and whenever $f : A \to B$ and $g : C \to D$ in $\mathbf{C}$, $\hom(f, g)$ is the set of pairs (h, k) with $h : A \to C$ and $k : B \to D$ such that the diagram



is commutative. The hom sets may not be disjoint, but as I said, you can make them disjoint with the use of tags. If $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$ when possible and $1_{(A,B)} = (1_A, 1_B)$, this is also a category, but a bit more interesting.

10. Let $\mathbf{C}$ be an arbitrary category, and define $\mathbf{C}^{\mathrm{op}}$ as follows: $\mathrm{ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{ob}(\mathbf{C})$; whenever $A, B \in \mathrm{ob}(\mathbf{C})$, $\hom_{\mathbf{C}^{\mathrm{op}}}(A, B) = \hom_{\mathbf{C}}(B, A)$; if $f : A \to B$ and $g : B \to C$ in $\mathbf{C}^{\mathrm{op}}$, define $gf : A \to C$ to be fg as given by $\mathbf{C}$, and 1_A in $\mathbf{C}^{\mathrm{op}}$ the same as that in $\mathbf{C}$. This is clearly a category; it is called the **dual category** of $\mathbf{C}$.

11. Let A be an arbitrary object of a category $\mathbf{C}$. Define $\mathbf{C}/A$ as follows: $\mathrm{ob}(\mathbf{C}/A)$ is the class of pairs of the form (B, f) where $B \in \mathrm{ob}(\mathbf{C})$ and $f : B \to A$. For $(B, f), (C, g) \in \mathrm{ob}(\mathbf{C}/A)$, $\hom((B, f), (C, g))$ is the set of morphisms $u : B \to C$ such that $f = gu$, that is,



is commutative. Tag the morphisms to make the hom sets disjoint. Then it is easy to see that $\mathbf{C}/A$ becomes a category by defining composition of morphisms and identity morphisms to agree with $\mathbf{C}$. $\mathbf{C}/A$ is called the **category of objects in C below** A.

12. Likewise, define $\mathbf{C}\backslash A$ by agreeing that $\mathrm{ob}(\mathbf{C}\backslash A)$ is the class of pairs of the form (B, f) where $f : A \to B$, and $\hom((B, f), (C, g))$ is the set of morphisms $u : B \to C$ such that $uf = g$. $\mathbf{C}\backslash A$ is then a category, called **the category of objects in C above** A.

Note that $\mathrm{ob}(\mathbf{C})$ is a *class*. It may not be a set, as we now see.

$\mathcal{S} = \mathrm{ob}(\mathbf{Set})$ is supposed to be the class of all sets. If $\mathcal{S}$ were a set, we could legally form $X = \{x \in \mathcal{S} \mid x \notin x\}$ [since every $x \in \mathcal{S}$ is a set, $a \in x$ is a defined statement]. But then $X \in X$ if and only if $X \notin X$, so this is an impossible situation. Hence, $\mathcal{S}$ cannot be a set without resulting in paradoxes. So $\mathcal{S}$ is said to be a *proper class*.

In rare occasions, such as in examples 6 and 7 above, **ob(C)** *is* a set. If **C** is a category such that ob(**C**) is a set, **C** is said to be a **small category**.

An **isomorphism** in a category is exactly what one would expect: $f : A \to B$ is an isomorphism provided that there exists $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$. In that case, g is unique, and is denoted f^{-1}.

Subcategories

Remarkably, it already follows that isomorphisms compose into isomorphisms. If $f : A \to B$ is an isomorphism with inverse f^{-1} and $g : B \to C$ is an isomorphism with inverse g^{-1}, then $gf : A \to C$ is an isomorphism with inverse $f^{-1}g^{-1} : C \to A$. And of course, 1_A is an isomorphism. This illustrates the following definition:

DEFINITION

A category **D** *is said to be a* ***subcategory*** *of a category* **C** *provided that:*

(1) *ob*(**D**) *is a subclass of ob*(**C**).

(2) *Whenever* $A, B \in ob(\mathbf{D})$, $\hom_{\mathbf{D}}(A, B)$ *is a subset of* $\hom_{\mathbf{C}}(A, B)$.

(3) *Whenever* $f : A \to B$ *and* $g : B \to C$ *in* **D**, *the composite* $gf : A \to C$ *given by* **C** *is in* $\hom_{\mathbf{D}}(A, C)$ *and is the composite* gf *given by* **D**.

(4) *For each* $A \in \mathbf{D}$, **C***'s identity morphism* 1_A *is in* $\hom_{\mathbf{D}}(A, A)$ *as* **D***'s identity morphism.*

If also $\hom_{\mathbf{D}}(A, B) = \hom_{\mathbf{C}}(A, B)$ *for all* $A, B \in ob(\mathbf{D})$, **D** *is said to be a* ***full subcategory*** *of* **C**.

Notice that subcategories of **C** can be identified purely in terms of their objects and morphisms, because **C** already gives the rest of the structure. And full subcategories can be identified from just the objects! They hypothetically leave all morphisms that they can.

EXAMPLES

1. Since every group is a monoid, no two groups can be the same monoid and every group homomorphism is a homomorphism of the monoids, **Grp** is a subcategory of **Mon**. However, **Mon** is *not* a subcategory of **Set**, because a set can be many different monoids. **Mon** *is* a subcategory of **Semgrp** [the semigroups] because a semigroup can't have more than one identity element.

2. Every monoid homomorphism of groups is automatically a group homomorphism, so **Grp** is a full subcategory of **Mon**. However, there exist maps of monoids preserving multiplication which don't map 1 to 1, hence **Mon** is *not* a full subcategory of **Semgrp**.

3. Since a rng can be at most one ring with the same addition and multiplication, **Ring** is a subcategory of **Rng**.

4. Since the isomorphisms in a category **C** are closed under defined composition and involve all identity morphisms, one can form a subcategory of **C** by keeping precisely the isomorphisms.

EXERCISES

1. If F and G are fields and $f : F \to G$ is a homomorphism, then f is injective. [*Hint*: What are the ideals in a field?]

2. (a) Suppose $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ are varieties, where every operator for $\mathcal{V}(S_2)$ is in $\mathcal{V}(S_1)$ and every identity in S_2 is in S_1. Then every $\mathcal{V}(S_1)$ algebra is also a $\mathcal{V}(S_2)$ algebra, and homomorphisms between $\mathcal{V}(S_1)$ algebras are also homomorphisms with $\mathcal{V}(S_2)$'s structure. Assume no two $\mathcal{V}(S_1)$ algebras can have the same $\mathcal{V}(S_2)$ structure. Show that $\mathbf{V}(S_1)$ is a subcategory of $\mathbf{V}(S_2)$.

   (b) If $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ have exactly the same operators, then $\mathbf{V}(S_1)$ is a full subcategory of $\mathbf{V}(S_2)$.

   (c) Show by example that $\mathcal{V}(S_1)$ may have operators that $\mathcal{V}(S_2)$ doesn't, but $\mathbf{V}(S_1)$ is still a full subcategory of $\mathbf{V}(S_2)$.

3. (a) An object I in a category **C** is said to be **initial** provided that for every object A, there is exactly one morphism in $\hom(I, A)$. For example, $I_S(\Omega)$ is initial in $\mathbf{V}(S)$ [see Exercise 10 of Section 1.9] and the King variety is initial in **Var** [see Exercise 4 of Section 1.11]. Show that any two initial objects in a category are isomorphic.

   (b) An object T is said to be **terminal** provided that for every object A, there is exactly one morphism in $\hom(A, T)$. For example, $T(\Omega)$ is terminal in $\mathbf{V}(S)$, and the variety of sets is terminal in **Var**. Explain why any two terminal objects in a category are isomorphic.

   (c) Whenever A is an object in a category **C**, $\mathbf{C}/A$ has a terminal object, and $\mathbf{C}\backslash A$ has an initial object. [*Hint*: Try $(A, 1_A)$.]

   (d) A **zero object** in a category **C** is an object which is both initial and terminal. If **C** has a zero object, show that one can assign each pair (A, B) of objects in **C** a morphism $0_{A,B} \in \hom(A, B)$ such that $0_{D,B}g = 0_{A,B} = f0_{A,C}$ when they are defined. In particular, show $\hom(A, B)$ is never empty for any objects $A, B \in \mathrm{ob}(\mathbf{C})$.

4. (a) **C** is a full subcategory for **C**, for every category **C**.

   (b) If **E** is a subcategory of **D** and **D** is a subcategory of **C**, then **E** is a subcategory of **C**.

   (c) Prove part (b) with "subcategory" replaced with "full subcategory."

5. A category **C** is discrete if and only if every subcategory of **C** is a full subcategory.

6. Is **Ring** is a full subcategory of **Rng**?

7. (a) Let $\mathcal{V}(S)$ be a variety, and form a category $\mathbf{V}(S)-\mathbf{sub}$ as follows: the objects are the pairs of the form (A, A_1) with $A \in \mathcal{V}(S)$, A_1 a subalgebra of A, and $\hom((A, A_1), (B, B_1))$ consists of homomorphisms $f : A \to B$ such that $f(A_1) \subseteq B_1$. Verify that this data forms a category with the usual composition of morphisms and identity morphisms.

   (b) Likewise, define $\mathbf{V}(S)-\mathbf{con}$ as follows: the objects are the pairs of the form (A, Φ) with $A \in \mathcal{V}(S)$, Φ a congruence relation on A, and $\hom((A, \Phi), (B, \Theta))$ consists of homomorphisms $f : A \to B$ such that whenever $a\Phi b$ in A, $f(a)\Theta f(b)$ in B. Then $\mathbf{V}(S)-\mathbf{con}$ is a category.

8. A small category $\mathbf{C}$ in which all morphisms are isomorphisms is called a **groupoid**. In this exercise we establish a non-categorial definition of a groupoid. We see it as a set G equipped with the following structure:

   (1) Whenever $a, b \in G$, ab is either some element of G or is *undefined*. [This is a **partial operator**.]

   (2) Whenever $a \in G$, a^{-1} is some element of G.

   (3) [associativity] Whenever ab and bc are defined in G, then $(ab)c$ and $a(bc)$ are defined and $(ab)c = a(bc)$.

   (4) [inverse] aa^{-1} and $a^{-1}a$ are always defined for $a \in G$.

   (5) [identity] Whenever ab is defined in G, $abb^{-1} = a$ and $a^{-1}ab = b$. [Note that rules (3) and (4) already show that those expressions are defined and unambiguous.]

   (a) If $\mathbf{C}$ is a groupoid, consider $G = \biguplus_{A,B\in \mathrm{ob}(\mathbf{C})} \hom(A, B)$. If $a : A \to B$ and $b : A' \to B'$ are in G, define ab to be $ab : A' \to B$ as given in $\mathbf{C}$ if $A = B'$, and undefined if $A \neq B'$. Then define a^{-1} to be $a^{-1} : B \to A$ as given in $\mathbf{C}$. Verify rules (3), (4), (5) for G.

   Now suppose G is any set equipped with structure satisfying the five rules above. Show that for $a, b \in G$:

   (b) $(a^{-1})^{-1} = a$. [*Hint*: Why is $(a^{-1})^{-1}a^{-1}a$ defined? Change it in two ways.]

   (c) If ab is defined, then $b^{-1}a^{-1}$ is defined and $b^{-1}a^{-1} = (ab)^{-1}$. [*Hint*: $b^{-1}a^{-1}ab$ and $ab(ab)^{-1}$ are defined [why?].]

   (d) For $a, b \in G$, define $a\Phi b$ to mean that ab^{-1} is defined. Then Φ is an equivalence relation on G.

   (e) If $a \in G$, $T(a) = \{x \in G \mid xa \text{ is defined}\}$ is an equivalence class of Φ, which may be different from a's. [Yes, this means it must be nonempty.]

   (f) Define $\mathrm{ob}(\mathbf{C}) = G/\Phi$, the set of equivalence classes, and for $A, B \in G/\Phi$, $\hom(A, B) = \{a \in A \mid T(a) = B\}$. If $a \in \hom(A, B)$ and $b \in \hom(B, C)$, $ba \in \hom(A, C)$. Also, $a^{-1}a$ is the same for all $a \in A$, and is an identity morphism in $\hom(A, A)$. Conclude that $\mathbf{C}$ is a groupoid in the categorical sense.

(g) Take a look at the translations for a groupoid in (a) and (f). If you go through one of them and then the other, must you end up with the same thing you started with?