1.11 - Takeoffs and Universals

Nicholas McConnell

(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired. It may be referenced later though.

Now that varieties are characterized, one must ask how you can go from one to another. We have briefly discussed extension signatures in the first section, but this section generalizes the idea.

To begin with, every group G is a monoid, because it has an associative binary operation with an identity. That's not all the group needs, but it still has it. The monoid structure on G is part of the group structure, with some details missing. This structure takes off algebraically, in the sense that every group homomorphism of groups is also a monoid homomorphism of the monoids.

Also, every ring R is a rng when you disregard the identity element, and a ring homomorphism of rings is automatically a rng homomorphism. [Note that a rng homomorphism can have rings for the domain and codomain without being a ring homomorphism — take the zero map $\mathbb{Z} \to \mathbb{Z}$, for example. However, a monoid homomorphism of groups is always a group homomorphism.]

Another interesting idea is this: Let R be a fixed ring, U its group of units. If M is an R-module, then U acts on M by assigning ux for $u \in U, x \in M$ to ux given by the R-module structure for M. Thus every R-module is a U-action in such a way that every homomorphism of R-modules is a homomorphism of the U-actions.

However, suppose some mathematics alien assigns every group G a ring structure in a random way. Then it's nearly impossible for every group homomorphism to be a homomorphism of the rings. We're not interested in that idea.

In each of the preceding examples, we formed variety by leaving certain operations and forgetting others. But here's a more nontrivial example: If A is an associative algebra over a commutative ring R, A becomes a Lie algebra by defining $[a : b] = ab - ba$. This operation isn't freshly one of the associative algebra's operations, but it is derived from the structure. Every homomorphism of associative algebras is evidently a homomorphism of the Lie algebras, because $f(ab - ba) = f(ab) - f(ba) = f(a)f(b) - f(b)f(a)$.

So the basic idea is to take the operations in one signature and define them using expressions from the other, but that's not good enough. Remember that to be in a variety, an algebra need not only have operations, but it must also satisfy identities. If the identities aren't satisfied in the first legitimate algebra, tough luck changing the structure.

Here's the rigorous definition of a takeoff of varieties:

DEFINITION

*Let Ω_1, Ω_2 be signatures, $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$. A **takeoff** from $\mathcal{V}(S_1)$ to $\mathcal{V}(S_2)$ is a mathematical object T with the following structure:*

(1) *For each $\omega \in \Omega_2(n)$, $T\omega$ is some element of $F_{S_1}(\Omega_1, \{x_1, x_2, \dots x_n\})$.*

(2) *Suppose* $h : F(\Omega_2, X_0) \to F_{S_1}(\Omega_1, X_0)$ *is the set map sending each symbol in* X_0 *to itself in the codomain, and each expression* $(\omega a_1 a_2 \dots a_n) \in F(\Omega_2, X_0)$ *to* $\varphi(T\omega)$ *with* $\varphi : F_{S_1}(\Omega_1, \{x_1, x_2, \dots, x_n\}) \to F_{S_1}(\Omega_1, X_0)$ *the homomorphism satisfying* $\varphi(x_i) = h(a_i)$ *for* $1 \le i \le n$. *Then* $h(w_1) = h(w_2)$ *for all* $(w_1, w_2) \in S_2$.

To make this definition less confusing, recall that $F_{S_1}(\Omega_1, \{x_1, x_2, \dots x_n\})$ consists of congruence classes of Ω_1-exrpessions in $x_1, x_2, \dots x_n$ given by the identities in S_1. Hence, $T\omega$ is one of these congruence classes, giving an expression to define ω as done with the Lie algebra.

If $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is a takeoff and $A \in \mathcal{V}(S_1)$, then for each $\omega \in \Omega_2(n)$, $a_1, a_2, \dots a_n \in A$, define $(\omega a_1 a_2 \dots a_n)$ to equal $\varphi(T\omega)$ where

$$\varphi : F_{S_1}(\Omega_1, \{x_1, x_2, \dots, x_n\}) \to A$$

is the $\mathcal{V}(S_1)$ homomorphism sending $x_i \to a_i$. Taking $F_{S_1}(\Omega_1, X_0)$ for A, the map h in Condition (2) of the definition is then basically a homomorphism of Ω_2-algebras, and the condition says that the identities in S_2 are satisfied for $F_{S_1}(\Omega_1, X_0)$.

The fundamental theorem about takeoffs is this:

THEOREM 1.27 *Let* $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ *be a takeoff of varieties. Then:*

(1) *Every* $A \in \mathcal{V}(S_1)$ *becomes an algebra in* $\mathcal{V}(S_2)$ *when defined as above.*

(2) *Every* Ω_1*-homomorphism of algebras in* $\mathcal{V}(S_1)$ *is also an* Ω_2*-homomorphism of the algebras in* $\mathcal{V}(S_2)$ *[i.e. it preserves the operators in* Ω_2*].*

Remarkably, the converse of this theorem holds; see Exercise 3. The $\mathcal{V}(S_2)$ structure defined above is called the **derived structure** from the takeoff.

Proof of Theorem 1.27. (1) The definition above defines each $\omega \in \Omega_2(n)$ for A. We need only show that A satisfies every identity in S_2. Let $f : F(\Omega_2, X_0) \to A$ be a homomorphism, $h : F(\Omega_2, X_0) \to F_{S_1}(\Omega_1, X_0)$ be the homomorphism given by Condition (2) in the definition of a takeoff. Since $A \in \mathcal{V}(S_1)$, the map $X_0 \to A$ sending $x_i \to f(x_i)$ extends to a homomorphism $j : F_{S_1}(\Omega_1, X_0) \to A$, and evidently $jh = f$. If $(w_1, w_2) \in S_2$, $h(w_1) = h(w_2)$ by definition of a takeoff, hence $f(w_1) = jh(w_1) = jh(w_2) = f(w_2)$ and $(w_1, w_2) \in \ker f$. Therefore, $S \subseteq \ker f$ for every homomorphism $f : F(\Omega_2, X_0) \to A$, which means that $A \in \mathcal{V}(S_2)$.

(2) Suppose $A, B \in \mathcal{V}(S_1)$ and $f : A \to B$ is a Ω_1-homomorphism. Let $\omega \in \Omega_2(n)$, we wish to show that $f(\omega a_1 a_2 \dots a_n) = (\omega f(a_1) f(a_2) \dots f(a_n))$ for $a_1, a_2, \dots a_n \in A$. Let $\varphi_A : F_{S_1}(\Omega, \{x_1, x_2, \dots x_n\}) \to A$ be the homomorphism sending $x_i \to a_i$, and $\varphi_B : F_{S_1}(\Omega, \{x_1, x_2, \dots x_n\}) \to B$ the homomorphism sending $x_i \to f(a_i)$. Evidently $\varphi_B = f\varphi_A$ because $f\varphi_A$ sends $x_i \to f(a_i)$ and φ_B is unique for this property. However, by definition of ω in A and B,

$$\varphi_A(T\omega) = (\omega a_1 a_2 \dots a_n)$$

$$\varphi_B(T\omega) = (\omega f(a_1) f(a_2) \ldots f(a_n))$$

Applying f to the first of these,

$$f\varphi_A(T\omega) = f(\omega a_1 a_2 \ldots a_n)$$

Therefore, since $f\varphi_A = \varphi_B$, it follows that f preserves ω and is an Ω_2-homomorphism, completing the proof. ■

The takeoffs act a lot like homomorphisms of algebras in a single variety. The reader is left to do Exercises 1-7 and make discoveries.

Universals

The notion of a free algebra given by a set can be generalized to any takeoff. Recall the takeoff from rings to rngs, for instance; we shall find a fundamental ring enveloping any rng R. Define the ring $\overline{R} = \mathbb{Z} \times R$ as follows:

$$(n, r) + (n', r') = (n + n', r + r')$$

$$(n, r)(n', r') = (nn', nr' + n'r + rr')$$

$$0 = (0, 0), 1 = (1, 0), -(n, r) = (-n, -r)$$

Direct verification shows that $\overline{R}$ is a ring under these operations, and that $i : R \to \overline{R}$ given by $i(r) = (0, r)$ is a rng homomorphism.

Now suppose S is any *ring* and $f : R \to S$ is a rng homomorphism. Define $h : \overline{R} \to S$ by $j(n, r) = n1 + f(r)$. Then h is readily seen to be a *ring* homomorphism, and of course, $f = hi$. In fact, h is unique for this property, because if $f = h'i$ where h' is another ring homomorphism $\overline{R} \to S$,

$$h'(n, r) = h'(n(1, 0) + (0, r)) = h'(n1 + i(r)) = nh'(1) + h'i(r) = n1 + f(r)$$

Hence, $h' = h$.

Notice that even if R is already a ring, $\overline{R}$ may be larger than R. This is because of a basic property failed by the takeoff from rings to rngs; see Exercise 8. Summarizing this to any takeoff, we have:

DEFINITION

Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties. If $A \in \mathcal{V}(S_2)$, a ***universal $\mathcal{V}(S_1)$-algebra enveloping A for the takeoff*** *is a pair (U, i) where $U \in \mathcal{V}(S_1)$ and $i : A \to U$ is an Ω_2-homomorphism, such that whenever (U', f) is another pair with $U' \in \mathcal{V}(S_1)$ and $f : A \to U'$ an Ω_2-homomorphism, there exists a unique Ω_1-homomorphism $h : U \to U'$ such that $f = hi$.*

The foregoing example shows that $\overline{R}$ is a universal ring enveloping the rng R. Also, if $\mathcal{V}(S_2)$ is the variety of sets and T is the unique takeoff [Exercise 4(a)], the universal U is the free $\mathcal{V}(S_1)$-algebra given by the set A.

It can be shown that a universal is unique up to isomorphism, and in fact exists with a subtle and interesting recipe:

THEOREM 1.28 *Let* $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ *be a takeoff of varieties, and* $A \in \mathcal{V}(S_2)$*. If* (U, i) *and* (U', i') *are both universal* $\mathcal{V}(S_1)$*-algebras enveloping* A*, there exists a unique isomorphism* $\sigma : U \to U'$ *such that* $i' = \sigma i$*.*

Proof of Theorem 1.28. Since (U', i') is a pair with $U' \in \mathcal{V}(S_1)$ and $i' : A \to U'$ a Ω_2-homomorphism, but (U, i) is universal for this property, there exists a unique Ω_1-homomorphism $\sigma : U \to U'$ such that $i' = \sigma i$. Reversing the roles of (U', i') and (U, i) shows that since (U, i) has a property (U', i') is universal for, there is an Ω_1-homomorphism $\sigma' : U' \to U$ such that $i = \sigma' i'$. Furthermore, $\sigma'\sigma i = \sigma' i' = i$. Since (U, i) is universal though, 1_U is the *unique* homomorphism $U \to U$ such that $1_U i = i$, and hence, $\sigma'\sigma = 1_U$ by uniqueness. Likewise, $\sigma\sigma' = 1_{U'}$. Therefore, σ is an isomorphism with inverse σ'. ■

THEOREM 1.29 *Let* $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ *be a takeoff of varieties, and* $A \in \mathcal{V}(S_2)$*. Then there exists a universal* (U, i) *enveloping* A*.*

Proof of Theorem 1.29. Let $F = F_{S_1}(\Omega_1, A)$ where A is regarded as a *set*, and let $j : A \to F$ be the canonical set map into the free algebra. Then, let Θ be the congruence relation on F generated by tuples of the form

$$(j(\omega a_1 a_2 \dots a_n), (\omega j(a_1) j(a_2) \dots j(a_n)))$$

with $\omega \in \Omega_2(n)$ and $a_1, a_2, \dots a_n \in A$. [The latter of these expressions used the derived Ω_2-structure for F.] Finally, let $\pi : F \to F/\Theta$ the canonical epimorphism, $i = \pi j$. We claim that $(F/\Theta, i)$ is a universal enveloping A.

To begin with, $i : A \to F/\Theta$ is an Ω_2-homomorphism because whenever $\omega \in \Omega_2(n)$ and $a_1, a_2, \dots a_n \in A$, $j(\omega a_1 a_2 \dots a_n)\Theta(\omega j(a_1) j(a_2) \dots j(a_n))$, so that

$$\pi j(\omega a_1 a_2 \dots a_n) = \pi(\omega j(a_1) j(a_2) \dots j(a_n)) = (\omega \pi j(a_1) \pi j(a_2) \dots \pi j(a_n))$$

by virtue of π. Hence, $\pi j = i$ is a homomorphism.

Now suppose $B \in \mathcal{V}(S_1)$ and $f : A \to B$ is a Ω_2-homomorphism. Since $B \in \mathcal{V}(S_1)$, the set map f extends to an Ω_1-homomorphism $\overline{f} : F \to B$ such that $f = \overline{f} j$. It turns out that $\Theta \subseteq \ker \overline{f}$ because f is a homomorphism, and hence,

$$\overline{f} j(\omega a_1 a_2 \dots a_n) = f(\omega a_1 a_2 \dots a_n) = (\omega f(a_1) f(a_2) \dots f(a_n))$$

$$= (\omega \overline{f} j(a_1) \overline{f} j(a_2) \dots \overline{f} j(a_n)) = \overline{f}(\omega j(a_1) j(a_2) \dots j(a_n))$$

Thus $\ker \overline{f}$ contains all tuples of the form $(j(\omega a_1 a_2 \dots a_n), (\omega j(a_1) j(a_2) \dots j(a_n)))$, and hence, Θ since it's generated by those tuples. By Theorem 1.10, a homomorphism $h : F/\Theta \to B$ satisfying $\overline{f} = h\pi$ exists. Meanwhile, $f = hi$ since $f = \overline{f} j = h\pi j = hi$.

To show that h is unique, suppose $h' : F/\Theta \to B$ is also an Ω_1-homomorphism satisfying $f = h'i$. Then $f = h'i = h'\pi j = \overline{f}'j$, where $\overline{f}' = h'\pi$. However, $\overline{f}$ is the *unique* homomorphism $F \to B$ satisfying $f = \overline{f}j$, so that $\overline{f} = \overline{f}'$ by uniqueness. Finally, $h\pi = h'\pi$, and hence, $h = h'$ since π is surjective. Therefore, h is unique and the proof is concluded. ■

The proof of the existence of universals outlined a "recipe" for finding them: Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties, and $A \in \mathcal{V}(S_2)$.

1. First, take the free $\mathcal{V}(S_1)$-algebra F given by the set A. This induces a set map $A \to F$ with a universal mapping property.

2. To make that map a homomorphism, take each expression in F using one operator in Ω_2 as taken from Ω_1's operators, and identify that expression with its value in A, by factoring out the generated congruence relation. Make *no more identifications than that*, so the map is universal for all homomorphisms from A to any Ω_1-algebra.

3. Compose the set map in Step 1 with the canonical epimorphism $F \to F/\Theta$ and you get an Ω_2-homomorphism $A \to F/\Theta$. F/Θ with that map is the desired universal.

Remember, any two universals are isomorphic by Theorem 1.27, so don't expect to have different choices for the results.

To try the recipe, let M be a fixed monoid, N a submonoid of M. Then every M-action X becomes an N-action if we restrict the actors in the monoid, and this is clearly a takeoff. Now let X be an N-action; we wish to find the universal M-action enveloping X. Step 1 tells us to start with the free M-action given by X, which we know is $M \times X$, along with the map $x \to (1, x)$ from $X \to M \times X$.

To do Step 2, we must take each *expression* in $M \times X$ [given by the free M-action] resulting in scalarly multiplying a symbol in X by an element of N, and identify it with its actual value in the N-action X. If $x \in X$ and $n \in N$, the former of these is (n, x), whereas the latter is $nx \to (1, nx)$. So if Θ is the congruence relation on $M \times X$ generated by $\{((n, x), (1, nx)) \mid x \in X, n \in N\}$, then $(M \times X)/\Theta$ is the desired universal M-action. Step 3 tells us that the corresponding map $X \to (M \times X)/\Theta$ is given by $x \to \overline{(1, x)}$.

A more difficult example is the group enveloping a monoid M. Knowing what a free group is, one can easily apply Steps 1-3. The resulting group G consists of expressions of the form $[m_1]m_2^{-1}m_3m_4^{-1}\dots m_{n-1}^{-1}[m_n]$, where $m_i \neq 1$ and $m_i \neq m_{i+1}$ in the reduced case.

EXERCISES

1. Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties and $A \in \mathcal{V}(S_1)$.

   (a) If B is a subalgebra of the Ω_1-algebra A, then B is also a subalgebra of A as an Ω_2-algebra, and the Ω_2-subalgebra structure on B is the same as the derived structure from the Ω_1-subalgebra structure. [*Hint*: Consider the canonical monomorphism.]

(b) If Φ is a congruence relation on the Ω_1-algebra A, then Φ is also a congruence relation of A as an Ω_2-algebra, and the Ω_2-quotient structure on A/Φ is the same as the derived structure from the Ω_1-quotient structure. [*Hint*: Consider the canonical epimorphism.]

(c) If $\{A_\alpha\}$ is a collection of algebras in $\mathcal{V}(S_1)$, then the derived Ω_2-structure of ΠA_α is the same as the structure for the product of the A_α's each with the derived structure. [*Hint*: Consider the projections from the product.]

2. An **affine** is a set X with a ternary operator $a\bar{b}c$ and the identities $(a\bar{b}c)\bar{d}f = a\bar{b}(c\bar{d}f)$ and $a\bar{b}b = a = b\bar{b}a$. Show that a group G is an affine when defined by $a\bar{b}c = ab^{-1}c$, and that this is a takeoff from the groups to the affines.

3. Let $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ be varieties. Suppose every algebra in $\mathcal{V}(S_1)$ is given a structure for an algebra in $\mathcal{V}(S_2)$, such that an Ω_1-homomorphism of algebras in $\mathcal{V}(S_1)$ is also an Ω_2-homomorphism. For each $\omega \in \Omega_2(n)$, define $T\omega = (\omega x_1 x_2 \dots x_n) \in F_{S_1}(\Omega_1, \{x_1, x_2, \dots x_n\})$. Show that T is the unique takeoff $\mathcal{V}(S_1) \to \mathcal{V}(S_2)$ in which the ensuing structures are the derived structures.

4. (a) Let $\mathcal{S}$ be the variety of sets, i.e. Ω-algebras with no operations in Ω whatsoever. Then there is a unique takeoff $\mathcal{V} \to \mathcal{S}$ where $\mathcal{V}$ is any variety.

   (b) Let $\mathcal{K}$ be the variety given by one nullary operator ϵ and one identity, $x = (\epsilon)$. Convince yourself that every $\mathcal{K}$-algebra is the one-element set $\{(\epsilon)\}$. [It is called the "King variety", if you insist.] Show that there's a unique takeoff $\mathcal{K} \to \mathcal{V}$ where $\mathcal{V}$ is any variety.

5. Let $T_1 : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ and $T_2 : \mathcal{V}(S_2) \to \mathcal{V}(S_3)$ be takeoffs of varieties. Define the **composite takeoff** $T_2T_1 : \mathcal{V}(S_1) \to \mathcal{V}(S_3)$ as follows: For each $\omega \in \Omega_3(n)$, $T_2T_1\omega = \varphi(T_2\omega)$ where $\varphi : F_{S_2}(\Omega_2, \{x_1, x_2, \dots x_n\}) \to F_{S_1}(\Omega_1, \{x_1, x_2, \dots x_n\})$ is the Ω_2-homomorphism sending each $x_i \to x_i$ [it exists because $F_{S_1}(\Omega_1, \{x_1, x_2, \dots x_n\}) \in \mathcal{V}(S_2)$].

   (a) T_2T_1 is a takeoff from $\mathcal{V}(S_1)$ to $\mathcal{V}(S_3)$, and for every $A \in \mathcal{V}(S_1)$, the derived Ω_3-structure given by T_2 of A as an Ω_2-algebra given by T_1's derived structure is the same as the derived Ω_3-structure of A given by T_2T_1.

   (b) If $T_3 : \mathcal{V}(S_3) \to \mathcal{V}(S_4)$ is another takeoff, $(T_3T_2)T_1 = T_3(T_2T_1)$. [*Hint*: Exercise 3 may help.]

   (c) Define the **identity takeoff** $1_{\mathcal{V}(S_1)} : \mathcal{V}(S_1) \to \mathcal{V}(S_1)$ by $T\omega = (\omega x_1 x_2 \dots x_n)$ for $\omega \in \Omega_1(n)$. Then the derived structure for an Ω_1-algebra A given by $1_{\mathcal{V}(S_1)}$ is simply its original structure.

   (d) If $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is any takeoff, then $T1_{\mathcal{V}(S_1)} = T = 1_{\mathcal{V}(S_2)}T$.

6. A takeoff $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is said to be an **isomorphism** [of varieties] if there exists another takeoff $T^{-1} : \mathcal{V}(S_2) \to \mathcal{V}(S_1)$ [called the **inverse** of T] such that $T^{-1}T = 1_{\mathcal{V}(S_1)}$ and $TT^{-1} = 1_{\mathcal{V}(S_2)}$.

   (a) Isomorphism of varieties is an equivalence relation.

   (b) The variety of abelian groups is isomorphic to the variety of $\mathbb{Z}$-modules. [*Hint*: Show that an abelian group, written additively, has a unique $\mathbb{Z}$-module structure.]

   (c) Let $1 \leq k \leq n$ be fixed positive integers. Suppose $\mathcal{V}$ is a variety given by one n-ary operator ω, and one identity, $(\omega x_1 x_2 \dots x_n) = x_k$. Then *every* set map of $\mathcal{V}$-algebras is a homomorphism, and $\mathcal{V}$ is isomorphic to the variety of sets.

   (d) The variety of groups is isomorphic to the variety of pointed affines [that is, affines with a nullary operator for the base point]. [*Hint*: Treat the base point as the group's identity element.]

   (e) A **Boolean ring** is a ring R satisfying $x^2 = x$ for all $x \in R$. Show that R is commutative and $1 + 1 = 0$ in R. Then, show that the takeoff from Boolean rings to Boolean algebras given by

   $$a \vee b = a + b - ab, a \wedge b = ab, 1 = 1, 0 = 0, a' = 1 - a$$

   is an isomorphism with inverse

   $$a + b = (a \wedge b') \vee (a' \wedge b), ab = a \wedge b, 1 = 1, 0 = 0, -a = a$$

   from the Boolean algebras to the Boolean rings.

   (f) Informally, what can you say about isomorphic varieties?

7. An **automorphism** of a variety is an isomorphism from the variety to itself.

   (a) The automorphisms of a variety form a group under takeoff composition. [You may assume that they form a set.]

   (b) Give examples of automorphisms of order 2 of the variety of monoids, of groups, of rings, of lattices, and of Boolean algebras. [*Hint*: If M is a monoid, define M^{op} by reversing the operands, $a * b = ba$.]

8. A takeoff $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is said to be **full** if every Ω_2-homomorphism of algebras in $\mathcal{V}(S_1)$ [with derived structure] is an Ω_1-homomorphism. For example, the takeoff from groups to monoids is full, but the takeoff from rings to rngs is not, because a rng homomorphism of rings need not map 1 to 1.

   (a) The composition of full takeoffs is full, and every isomorphism of varieties is full.

   (b) If $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is a full takeoff and A is an Ω_1-algebra, then $(A, 1_A)$ is the universal enveloping A [as an Ω_2-algebra with the derived structure] for T.

   (c) Show by example that part (b) may be false if T is not full.

9. (a) Suppose $T_1 : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ and $T_2 : \mathcal{V}(S_2) \to \mathcal{V}(S_3)$ are takeoffs, and let $A \in \mathcal{V}(S_3)$. If (U_2, i_2) is a universal enveloping A for T_2, and (U_1, i_1) is a universal enveloping U_2 for T_1, then $(U_1, i_1 i_2)$ is a universal enveloping A for $T_2 T_1$.

   (b) If $T_1 : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is a takeoff, then a universal enveloping a free $\mathcal{V}(S_2)$-algebra given by a set X is a free $\mathcal{V}(S_1)$-algebra given by X. [*Hint*: Apply part (a) with $\mathcal{V}(S_3)$ the variety of sets.]

   (c) Under the takeoff from rings to monoids taking only the multiplication and 1 of a ring, the universal ring envelopping a monoid M is the direct sum $\sum_{m \in M} \mathbb{Z}$ with multiplication defined using the monoid operation and the distributive laws.

   (d) Using parts (b) and (c), figure out the free ring given by a set.

10. (a) Consider the takeoff from abelian groups to groups which forgets the commutativity requirement. The universal abelian group enveloping a group G is then $G/[G, G]$, where $[G, G]$ is the commutator subgroup of G.

    (b) Now consider the takeoff from commutative rings to rings. What's the universal commutative ring enveloping a ring R?

11. Let R be a fixed commutative ring, and consider the takeoff from associative algebras over R to Lie algebras over R given by $[a : b] = ab - ba$. Use the above recipe to find the universal associative algebra enveloping a Lie algebra L. [*Hint*: The free associative algebra given by a set is a bit similar to the free ring given by a set.]

12. Let $\mathcal{V}(S)$ be a variety. Define a new variety $\mathcal{V}(S')$ by adding a unary operator η which must be a homomorphism from the algebra to itself; that is, $(\eta(\omega a_1 a_2 \ldots a_n)) = (\omega(\eta a_1)(\eta a_2) \ldots (\eta a_n))$ for $\omega \in \Omega(n)$, $a_i \in A$. [This is called a $\mathcal{V}(S)$ **algebra with operator**.] Now consider the takeoff $\mathcal{V}(S') \to \mathcal{V}(S)$ which forgets the operator η and takes all other operators. Show that if $A \in \mathcal{V}(S)$, the universal $\mathcal{V}(S')$-algebra enveloping A is $\coprod_{n \in \mathbb{N}} A$, with η shifting up the operands of the coproduct.

13. Let $\mathcal{V}(S)$ be a variety. Define a new variety $\mathcal{V}(S')$ by adding a nullary operator ϵ for a base point. [This is called a **pointed** $\mathcal{V}(S)$ **algebra**.] Now consider the takeoff $\mathcal{V}(S') \to \mathcal{V}(S)$ which forgets the operator ϵ and takes all other operators. If $A \in \mathcal{V}(S)$, describe the universal $\mathcal{V}(S')$-algebra enveloping A.

14. Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff, $\{A_\alpha\}$ a family of $\mathcal{V}(S_2)$-algebras. For each α, let U_α be a universal Ω_1-algebra enveloping A_α. Show that the universal preserves coproducts: $\coprod U_\alpha$ is a universal Ω_1-algebra enveloping $\coprod A_\alpha$. Then determine the map. [*Caution*: $\coprod U_\alpha$ takes the coproduct in $\mathcal{V}(S_1)$, not in $\mathcal{V}(S_2)$.]