1.10 - Birkhoff's Theorem

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

An interesting thing about varieties is this: If you use identities to restrict the Ω-algebras in a class, the products and coproducts both exist, but the product stays the same, whereas the coproduct changes. Likewise, the terminal algebra $T(\Omega)$ stays as its 1-element self, but the initial algebra $I_S(\Omega)$ may be different. So we can claim that a variety $\mathcal{V}(S)$ can be closed under products, but "closed under coproducts" doesn't really make sense.

Let's have another look at the variety's properties in Theorem 1.21.

(1) $\mathcal{C}$ contains the terminal algebra $T(\Omega)$.
(2) If $A \in \mathcal{C}$, every subalgebra of A is in $\mathcal{C}$.
(3) If $A \in \mathcal{C}$, every homomorphic image of A is in $\mathcal{C}$.
(4) If $\{A_\alpha\}$ is a batch of [not necessarily distinct] algebras in $\mathcal{C}$, the product $\Pi A_\alpha \in \mathcal{C}$.

Our main goal is to show the converse of Theorem 1.21: every class $\mathcal{C}$ of Ω-algebras satisfying conditions (1)-(4) is a variety. This is known as the **HSP theorem** [homomorphic-image subalgebra product] in universal algebra. Note that (4) states $\mathcal{V}(S)$ is closed under *arbitrary* products — even if there are infinitely many factors. [Otherwise, the theorem would not hold; see Exercise 1.] We bring two statements from Section 6 here for our proof:

Fact 1. A subdirect product of A_α's is isomorphic to a subalgebra of their product. [definition]
Fact 2. If Φ_α's are congruence relations on A and $\Phi = \cap\Phi_\alpha$, then A/Φ is a subdirect product of the A/Φ_α's. [Exercise 1 of Section 6]

Now suppose $\mathcal{C}$ satisfies conditions (1)-(4). To begin with, condition (3) implies that $\mathcal{C}$ is closed under isomorphic copies.

Now let X be a set. If $A \in \mathcal{C}$ is nonempty, there exists a homomorphism $F(\Omega, X) \to A$. Let $\mathrm{Id}(X, A)$ be the intersection of all kernels of homomorphisms $F(\Omega, X) \to A$. Then $\mathrm{Id}(X, A)$ is the congruence relation consisting of the identities satisfied by A. We claim that $F(\Omega, X)/\mathrm{Id}(X, A) \in \mathcal{C}$. This is because for each homomorphism $f : F(\Omega, X) \to A$, surjectification and injectification together show that $F(\Omega, X)/\ker f \cong \mathrm{im}\, f$. $\mathrm{im}\, f$ is a subalgebra of A, hence is in $\mathcal{C}$ by condition (2). By closure under isomorphic copies, $F(\Omega, X)/\ker f \in \mathcal{C}$.

So $\mathcal{C}$ contains $F(\Omega, X)/\ker f$ for every homomorphism $f : F(\Omega, X) \to A$. Since $\mathrm{Id}(X, A)$ is the intersection of these $\ker f$'s, $F(\Omega, X)/\mathrm{Id}(X, A)$ is a subdirect product of the algebras of the form $F(\Omega, X)/\ker f$ by Fact 2. Since a subdirect product of algebras is a subalgebra of the product, conditions (2) and (4) show that $\mathcal{C}$ contains any subdirect product of algebras in $\mathcal{C}$. This is why $F(\Omega, X)/\mathrm{Id}(X, A) \in \mathcal{C}$.

Now let $\mathrm{Id}(X, \mathcal{C}) = \bigcap \mathrm{Id}(X, A)$ where the intersection is taken over all nonempty $A \in \mathcal{C}$ [there is at least one such algebra, namely $T(\Omega)$]. $\mathrm{Id}(X, \mathcal{C})$ is the set of identities satisfied by all algebras in $\mathcal{C}$ and is called the **congruence relation of identities for** $\mathcal{C}$. Since $F(\Omega, X)/\mathrm{Id}(X, A) \in \mathcal{C}$ for all nonempty $A \in \mathcal{C}$, the closure of $\mathcal{C}$ under subdirect products [seen in the previous paragraph] implies $F(\Omega, X)/\mathrm{Id}(X, \mathcal{C}) \in \mathcal{C}$. We claim this:

LEMMA 1.25 *If $\mathcal{C}$ is a class of Ω-algebras satisfying conditions (1)-(4) in Theorem 1.21, and X is a set, $F(\Omega, X)/Id(X, \mathcal{C})$ along with the map $i : X \to F(\Omega, X)/Id(X, \mathcal{C})$ sending $x \to \overline{x}$ is a free algebra for $\mathcal{C}$ given by X.*

Proof of Lemma 1.25. Assume π refers to the canonical epimorphism $F(\Omega, X) \to F(\Omega, X)/\mathrm{Id}(X, \mathcal{C})$, and $j : X \to F(\Omega, X)$ is the usual injection into the free algebra. Then $i = \pi j$.

Let $A \in \mathcal{C}$ and $f : X \to A$ a set map. f extends to a homomorphism $g : F(\Omega, X) \to A$ with $gj = f$. Furthermore, $\mathrm{Id}(X, \mathcal{C}) \subseteq \mathrm{Id}(X, A) \subseteq \ker g$ [since $\mathrm{Id}(X, \mathcal{C})$ is the intersection of the $\mathrm{Id}(X, A)$'s, and similarly for each $\mathrm{Id}(X, A)$], so g can be injectified to $\overline{g} : F(\Omega, X)/\mathrm{Id}(X, \mathcal{C}) \to A$ with $\overline{g}\pi = g$. We see that $\overline{g}i = \overline{g}\pi j = gj = f$.

Now suppose $\overline{g}' : F(\Omega, X)/\mathrm{Id}(X, \mathcal{C}) \to A$ also satisfies $\overline{g}'i = f$. Take $g' = \overline{g}'\pi$; then $g'j = \overline{g}'\pi j = \overline{g}'i = f$. But g is the *unique* homomorphism $F(\Omega, X) \to A$ satisfying $gj = f$ [since $F(\Omega, X)$ is free in the class of all Ω-algebras], hence $g = g'$. Furthermore, $\overline{g}\pi = \overline{g}'\pi$ and $\overline{g} = \overline{g}'$ since π is surjective. Therefore $\overline{g}$ is unique, and $(F(\Omega, X)/\mathrm{Id}(X, \mathcal{C}), i)$ constitutes a free algebra for $\mathcal{C}$ given by X. ■

Now here's our main result!

THEOREM 1.26 (BIRKHOFF'S THEOREM) *A class $\mathcal{C}$ of Ω-algebras is a variety if and only if it satisfies conditions (1)-(4) above.*

Proof of Theorem 1.26. If $\mathcal{C}$ is a variety, it satisfies conditions (1)-(4) by Theorem 1.21. Conversely, suppose $\mathcal{C}$ is a class of Ω-algebras satisfying conditions (1)-(4). Let $S = \mathrm{Id}(X_0, \mathcal{C})$. Then S is the set of identities satisfied by every algebra in $\mathcal{C}$; furthermore, $\mathcal{C}$ is contained in the variety $\mathcal{V}(S)$. We show that $\mathcal{V}(S) \subseteq \mathcal{C}$, so that $\mathcal{C} = \mathcal{V}(S)$ is a variety.

Suppose $A \in \mathcal{V}(S)$. Let $X \subseteq A$ be a set of generators of A. The injection map $X \to A$ extends to a homomorphism $f : F(\Omega, X) \to A$ sending each expression in X to its value in A. Since $X \subseteq \mathrm{im}\ f$ and generates A, f is surjective. We claim that $\mathrm{Id}(X, \mathcal{C}) \subseteq \ker f$: let $(w_1, w_2) \in \mathrm{Id}(X, \mathcal{C})$. By Lemma 1.20(2), X has a finite subset X' such that w_1, w_2 are in $F(\Omega, X')$. Exercise 2 shows that there exist maps $\lambda : X \to X_0$, $\zeta : X_0 \to X$ satisfying $\zeta\lambda(x) = x$ for all $x \in X'$. The map $i\zeta : X_0 \to F(\Omega, X)$ extends to a homomorphism $\zeta_1 : F(\Omega, X_0) \to F(\Omega, X)$ sending $x \in X_0$ to $\zeta(x)$. Likewise, λ extends to a homomorphism $\lambda_1 : F(\Omega, X) \to F(\Omega, X_0)$ sending $x \in X$ to $\lambda(x)$. Furthermore, $\zeta_1\lambda_1(x) = x$ when x is a symbol in $F(\Omega, X)$ from X'. Since those symbols generate $F(\Omega, X')$, $\zeta_1\lambda_1$ fix all elements of $F(\Omega, X')$, in particular, w_1 and w_2. Thus $\zeta_1\lambda_1(w_1) = w_1$ and $\zeta_1\lambda_1(w_2) = w_2$. Consider $f\zeta_1 : F(\Omega, X_0) \to A$. Since $A \in \mathcal{V}(S)$, $S = \mathrm{Id}(X_0, \mathcal{C})$ is contained in the kernel of $f\zeta_1$, so it can be injectified into a homomorphism $f_1 : F(\Omega, X_0)/\mathrm{Id}(X_0, \mathcal{C}) \to A$ such that $f_1\pi = f\zeta_1$, with π the canonical epimorphism as usual. Since $F(\Omega, X_0)/\mathrm{Id}(X_0, \mathcal{C}) \in \mathcal{C}$ though [by the discussion preceding Lemma 1.25], $\pi\lambda_1 : F(\Omega, X) \to F(\Omega, X_0)/\mathrm{Id}(X_0, \mathcal{C})$ has kernel containing $\mathrm{Id}(X, \mathcal{C})$. In particular, $\pi\lambda_1(w_1) = \pi\lambda_1(w_2)$. Furthermore, $f(w_1) = f\zeta_1\lambda_1(w_1) = f_1\pi\lambda_1(w_1) = f_1\pi\lambda_1(w_2) = f\zeta_1\lambda_1(w_2) = f(w_2)$ and $(w_1, w_2) \in \ker f$. Therefore, $\mathrm{Id}(X, \mathcal{C}) \subseteq \ker f$.

As a consequence, f can be injectified into a homomorphism which maps $F(\Omega, X)/\mathrm{Id}(X, \mathcal{C}) \to A$ by Theorem 1.10, which is surjective because f is. Therefore, A is a homomorphic image of $F(\Omega, X)/\mathrm{Id}(X, \mathcal{C})$. By the discussion preceding Lemma 1.25, $F(\Omega, X)/\mathrm{Id}(X, \mathcal{C}) \in \mathcal{C}$; hence $A \in \mathcal{C}$ by condition (3). Therefore, $\mathcal{C} = \mathcal{V}(S)$. ■

Recall Theorem 1.24: in the variety, coproducts always exist and are unique up to isomorphism. The conclusion is that if conditions (1)-(4) are satisfied, there is a "slight closure" under coproducts: any indexed collection of Ω-algebras in $\mathcal{V}(S)$ have *some coproduct in* $\mathcal{V}(S)$ *unique up to isomorphism*, but it may not be isomorphic to their coproduct in all Ω-algebras.

Likewise, $\mathcal{V}(S)$ contains an initial algebra $I_S(\Omega)$, but it doesn't have the same meaning as $I(\Omega)$. It turns out that since $I_S(\Omega)$ has no subalgebra except itself, it's a homomorphic image of $I(\Omega)$ by Exercise 9(d) of Section 9.

EXERCISES

1. If a class of Ω-algebras satisfies conditions (1)-(3) of a variety and is closed under *finite* products, show by example that it need not be a variety.

2. If $X_0 = \{x_0, x_1, \dots\}$, X is a set and X' a finite subset of X, show without using the Axiom of Choice that there exist maps $\lambda : X \to X_0$, $\zeta : X_0 \to X$ satisfying $\zeta\lambda(x) = x$ for all $x \in X'$.

3. (YONEDA'S THEORY) Let $\mathcal{C}$ be a class of Ω-algebras [which may not be a variety]. For $A, B \in \mathcal{C}$, let $\mathrm{hom}(A, B)$ denote the set of homomorphisms from A to B. Fix $A \in \mathcal{C}$. A **natural transformation** for A is an object η which assigns each $B \in \mathcal{C}$ a map $\eta_B : \mathrm{hom}(A, B) \to B$ satisfying

$$\eta_{B'}(kf) = k(\eta_B(f))$$

for all homomorphisms $f : A \to B$, $k : B \to B'$.

(a) If $\omega \in \Omega(0)$, then $[\omega]$ is a natural transformation for A when defined by $[\omega]_B(f) = (\omega_B)$ whenever $B \in \mathcal{C}, f : A \to B$.

(b) If $n \geq 1$, $\omega \in \Omega(n)$ and $\eta^1, \eta^2, \dots \eta^n$ are natural transformations for A, then $(\omega\eta^1\eta^2 \dots \eta^n)$ given by

$$(\omega\eta^1\eta^2 \dots \eta^n)_B(f) = (\omega\eta^1_B(f)\eta^2_B(f) \dots \eta^n_B(f))$$

for $B \in \mathcal{C}, f : A \to B$ is also a natural transformation for A.

We have established an Ω-algebra structure for the set N of natural transformations for A. We show that N is actually isomorphic to A.

(c) If $a \in A$, define $[a]_B(f) = f(a)$ for $B \in \mathcal{C}, f : A \to B$. Then $[a]$ is a natural transformation for A.

(d) The map $\varphi : A \to N$ sending $a \to [a]$ is an isomorphism, whose inverse is the map $\varphi^{-1} : N \to A$ sending $\eta \to \eta_A(1_A)$.