1.9 - Varieties and Coproducts

Nicholas McConnell

(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

We finally have enough tools to deal with the presence of identities! Recall that an identity indicates what expressions in variables must be equal for all substitutions. Expressions are, though, elements of $F(\Omega, X)$, which leads to the following definition.

DEFINITION
*A pair $(w_1, w_2) \in F(\Omega, X)^2$ is called an **identity** for Ω. An Ω-algebra A is said to **satisfy** the identity (w_1, w_2) if $f(w_1) = f(w_2)$ for every homomorphism $f : F(\Omega, X) \to A$.*

For the rest of this chapter, we let X_0 be the countably infinite set $\{x_0, x_1, x_2, \ldots\}$. For example, suppose Ω has a single binary operation [written multiplicatively] and A is an Ω-algebra. Then $((x_0x_1)x_2, x_0(x_1x_2))$ is an identity. What does it mean to say that A satisfies that identity? Well, suppose that for every homomorphism $f : F(\Omega, X_0) \to A$, $f((x_0x_1)x_2) = f(x_0(x_1x_2))$. This says $(f(x_0)f(x_1))f(x_2) = f(x_0)(f(x_1)f(x_2))$ for every homomorphism $f : F(\Omega, X_0) \to A$. Since there exists a homomorphism $F(\Omega, X_0) \to A$ with any given action on the $x_0, x_1, \ldots$, it turns out that $(ab)c = a(bc)$ for *all* $a, b, c \in A$. The argument can be traced both ways. Hence, A satisfies $((x_0x_1)x_2, x_0(x_1x_2))$ if and only if $(ab)c = a(bc)$ for all $a, b, c \in A$. It all makes sense! An identity (w_1, w_2) can sometimes be referred to as $w_1 = w_2$.

This rigorates the definition of a monoid: suppose $\Omega(0) = \{1\}$ and $\Omega(2) = \{p\}$. Then an Ω-algebra A is a monoid if and only if it satisfies the identities:

1. $((p(px_0x_1)x_2), (px_0(px_1x_2)))$ [associativity];
2. $((p(1)x_0), x_0)$ [left identity];
3. $((px_0(1)), x_0)$ [right identity].

These can be rewritten as follows: $(x_0x_1)x_2 = x_0(x_1x_2)$, $1x_0 = x_0$, $x_01 = x_0$.

To generalize the idea, suppose $S \subseteq F(\Omega, X_0)^2$ is a set of identities. Let $\mathcal{V}(S)$ be the class of all Ω-algebras satisfying every identity in S. Then $\mathcal{V}(S)$ is called a **variety**. For example, the groups form a variety, as do the rings [Exercises 1 and 2]. $\mathcal{V}(S)$ has some interesting closure properties, as we now see. If $X' \subseteq X$, recall how $F(\Omega, X')$ is a subalgebra of $F(\Omega, X)$ from Exercise 4 of Section 8.

LEMMA 1.20 (1) *For each $w \in F(\Omega, X)$, there exists a finite subset X' of X such that $w \in F(\Omega, X')$.*

(2) *If $w_1, w_2 \in F(\Omega, X)$, there exists a finite subset X' of X such that $F(\Omega, X')$ contains both w_1 and w_2.*

Be careful: (1) does *not* imply that there's a finite subset $X' \subseteq X$ such that $F(\Omega, X') = F(\Omega, X)$! It says that each $w \in F(\Omega, X)$ is in $F(\Omega, X')$ for some finite subset X' of X. The finite subsets, no matter how chosen, are widely different depending on which $w \in F(\Omega, X)$ we deal with. You can picture the theorem in one sentence: *expressions and identities are finite.* They only use finitely many symbols, due to the notion of length.

Proof of Lemma 1.20. (1) Let A be the set of $w \in F(\Omega, X)$ with the property stated in (1). We claim that $A = F(\Omega, X)$. To show this, we show that A is a subalgebra of $F(\Omega, X)$ containing $i(X)$. Each $i(x) \in i(X)$ is in $F(\Omega, \{x\})$ and $\{x\}$ is a finite subset of X, so every element of $i(X)$ satisfies the property and $i(X) \subseteq A$. If $\omega \in \Omega(0)$, then $(\omega) \in F(\Omega, \emptyset)$ [because it's in every subalgebra of $F(\Omega, X)$] and $\emptyset$ is finite, so $(\omega) \in A$. Now suppose $n \geq 1$, $\omega \in \Omega(n)$ and $a_1, a_2, \ldots a_n \in A$. Then there are finite subsets $X_1, X_2, \ldots X_n \subseteq X$ such that $a_i \in F(\Omega, X_i)$ for every i. The union $U = X_1 \cup X_2 \cup \cdots \cup X_n$ is a finite subset of X and $a_i \in F(\Omega, U)$ for every i. Hence, $(\omega a_1 a_2 \ldots a_n) \in F(\Omega, U)$, which means $(\omega a_1 a_2 \ldots a_n)$ satisfies the property and is in A. Hence, A is a subalgebra of $F(\Omega, X)$ containing $i(X)$, and is therefore $F(\Omega, X)$ since the algebra is generated by $i(X)$.

(2) If $w_1, w_2 \in F(\Omega, X)$, there exist finite subsets $X_1, X_2 \subseteq X$ with $w_i \in F(\Omega, X_i)$ for $i = 1, 2$ by part (1). $X_1 \cup X_2$ is a finite subset of X and $F(\Omega, X_1 \cup X_2)$ contains both w_1 and w_2. ■

THEOREM 1.21 (1) *$\mathcal{V}(S)$ contains the terminal algebra $T(\Omega)$.*

(2) *If $A \in \mathcal{V}(S)$, every subalgebra of A is in $\mathcal{V}(S)$.*

(3) *If $A \in \mathcal{V}(S)$, every homomorphic image of A is in $\mathcal{V}(S)$.*

(4) *If $\{A_\alpha\}$ is a batch of [not necessarily distinct] algebras in $\mathcal{V}(S)$, the product $\Pi A_\alpha \in \mathcal{V}(S)$.*

Note that if $A \cong B$, the isomorphism $A \to B$ is surjective, and hence, B is a homomorphic image of A. So part (3) implies that every isomorphic copy of an algebra in $\mathcal{V}(S)$ is in $\mathcal{V}(S)$ — and you don't need to worry over relabeling elements of an algebra.

Also, the fields do *not* form a variety. For one thing, property (4) fails: the product of fields is not a field.

Proof of Theorem 1.21. (1) There is only one homomorphism $f : F(\Omega, X_0) \to T(\Omega)$ and it maps every expression to the unique element of $T(\Omega)$. Hence, $f(w_1) = f(w_2)$ for every $(w_1, w_2) \in S$, and $T(\Omega) \in \mathcal{V}(S)$.

(2) Suppose B is a subalgebra of A, $f : F(\Omega, X_0) \to B$ is a homomorphism and $(w_1, w_2) \in S$. If $\iota : B \hookrightarrow A$ is the canonical monomorphism, $\iota f : F(\Omega, X_0) \to A$ is a homomorphism, hence $\iota f(w_1) = \iota f(w_2)$ since $A \in \mathcal{V}(S)$. Therefore, $f(w_1) = f(w_2)$ since ι is injective. Consequently, $B \in \mathcal{V}(S)$.

(3) Let $\eta : A \to B$ be the surjective homomorphism which is hypothesized to exist, $f : F(\Omega, X_0) \to B$ a homomorphism and $(w_1, w_2) \in S$. By Lemma 1.20(2), there exists a finite set $X' \subseteq X_0$ such that $F(\Omega, X')$ contains w_1 and w_2. For each $x_k \in X'$, choose $a_k \in A$ so that $\eta(a_k) = f(x_k)$ [since η is surjective, this is possible; and no Axiom of Choice is needed since X' is finite]. Pick one random element of A [what does this proof become if $A = \emptyset$?] to be

a_k whenever $x_k \in X_0 - X'$. The map $x_k \to a_k$ from X_0 to A extends to a homomorphism $g : F(\Omega, X) \to A$. We know that $f(x_k) = \eta g(x_k)$ for all $x_k \in i(X')$, because $\eta g(x_k) = \eta(a_k) = f(x_k)$. Hence $f(w) = \eta g(w)$ for all $w \in F(\Omega, X')$ by Exercise 10(a) of Section 3. In particular, $f(w_1) = \eta g(w_1)$ and $f(w_2) = \eta g(w_2)$. But $g(w_1) = g(w_2)$, since $A \in \mathcal{V}(S)$, therefore, applying η to both sides, $f(w_1) = f(w_2)$. Therefore, B satisfies all identities in S and hence is in $\mathcal{V}(S)$.

(4) Suppose $f : F(\Omega, X_0) \to \Pi A_\alpha$ is a homomorphism and $(w_1, w_2) \in S$. For each α, recall the projection $p_\alpha : \Pi A_\alpha \to A_\alpha$ and consider $p_\alpha f : F(\Omega, X_0) \to A_\alpha$. Since $A_\alpha \in \mathcal{V}(S)$, $p_\alpha f(w_1) = p_\alpha f(w_2)$. Hence, $f(w_1)_\alpha = f(w_2)_\alpha$ for all indices α, so that $f(w_1) = f(w_2)$. It follows that $\Pi A_\alpha \in \mathcal{V}(S)$. ■

Do you realize what we've done? We've just given a general proof that applies to monoids, groups, rings, lattices, Boolean algebras, R-modules for a fixed ring R, and so much more! Don't get overpumped; next chapter will be even better!

You probably asked whether free algebras exist in $\mathcal{V}(S)$. They certainly do, and we take the following approach to find them. If X is a set, define $\Phi(X, S)$ to be the congruence relation on $F(\Omega, X)$ generated by the set

$$\{(\varphi(w_1), \varphi(w_2)) \mid (w_1, w_2) \in S, \varphi \text{ a homomorphism } F(\Omega, X_0) \to F(\Omega, X)\}$$

Note that we took all *images* of the identities. For example, the distributive law $a(b + c) = ab + ac$ in a ring, after substituting into b the expression $x + yz$, yields $a((x + yz) + c) = a(x + yz) + ac$, and that must hold in a ring. By closing the relation into a congruence, we also regarded complicated results like $1x + (ab)c = x + a(bc)$.

Now put $F_S(\Omega, X) = F(\Omega, X)/\Phi(X, S)$. We show:

THEOREM 1.22 *The Ω-algebra $F_S(\Omega, X)$ along with the map $j : X \to F_S(\Omega, X)$ sending $x \to \overline{x}$ is a free algebra for $\mathcal{V}(S)$ given by X.*

Proof of Theorem 1.22. First we show that $F_S(\Omega, X) \in \mathcal{V}(S)$. Suppose $f : F(\Omega, X_0) \to F_S(\Omega, X)$ is a homomorphism and $(w_1, w_2) \in S$. X has a finite subset X' such that $w_1, w_2 \in F(\Omega, X')$ by Lemma 1.20(2). For each $x_k \in X'$, choose $a_k \in F(\Omega, X)$ so that $\overline{a_k} = f(x_k)$. Pick one random element of $F(\Omega, X)$ to be a_k for each $x_k \in X_0 - X'$. The map $x_k \to a_k$ from X_0 to $F(\Omega, X)$ extends to a homomorphism $g : F(\Omega, X_0) \to F(\Omega, X)$. Notice that if $\pi : F(\Omega, X) \to F_S(\Omega, X)$ is the canonical epimorphism, $f(x_k) = \pi g(x_k)$ for $x_k \in X'$, and hence, $f(w_1) = \pi g(w_1)$ and $f(w_2) = \pi g(w_2)$ by Exercise 10(a) of Section 3. However, $(g(w_1), g(w_2)) \in \Phi(X, S)$ by definition, whence $\pi g(w_1) = \pi g(w_2)$, since $\Phi(X, S)$ is the kernel of π. Furthermore, $f(w_1) = f(w_2)$, so that $F_S(\Omega, X) \in \mathcal{V}(S)$.

Now let $A \in \mathcal{V}(S)$ and $f : X \to A$ a set map. This yields an Ω-algebra homomorphism $f_1 : F(\Omega, X) \to A$ such that $f_1 j = f$. We claim that $\Phi(X, S) \subseteq \ker f_1$: to show this, we need only show that $(\varphi(w_1), \varphi(w_2)) \in \ker f_1$ whenever $(w_1, w_2) \in S$ and $\varphi : F(\Omega, X_0) \to F(\Omega, X)$ is a homomorphism. This is because $\Phi(X, S)$ is generated by the pairs of that form, hence *any* congruence relation containing them — in particular, $\ker f_1$ — contains $\Phi(X,S)$. The claim follows from $f_1\varphi$ being a homomorphism $F(\Omega, X_0) \to A$; hence $f_1\varphi(w_1) = f_1\varphi(w_2)$ since $(w_1, w_2) \in S$ and $A \in \mathcal{V}(S)$. Thus $(\varphi(w_1), \varphi(w_2))$ is in the kernel of f_1. Therefore, $\Phi(X,S) \subseteq \ker f_1$.

By Theorem 1.10, there is a homomorphism $\overline{f}_1 : F_S(\Omega, X) \to A$ such that $\overline{f}_1\pi = f_1$ with π the canonical epimorphism. Also, notice that $j = \pi i$. Hence $\overline{f}_1 j = \overline{f}_1\pi i = f_1 i = f$.

Now suppose $\overline{f}'_1 : F_S(\Omega, X) \to A$ is also a homomorphism satisfying $\overline{f}'_1 j = f$. Put $f'_1 = \overline{f}'_1\pi$. Then $f'_1 i = \overline{f}'_1\pi i = \overline{f}'_1 j = f$. But f_1 is the *unique* homomorphism $F(\Omega, X) \to A$ such that $f_1 j = f$, so we must have $f_1 = f'_1$. Hence $\overline{f}_1\pi = \overline{f}'_1\pi$. Since π is surjective, $\overline{f}_1 = \overline{f}'_1$ follows, and $\overline{f}'_1$ is unique. ■

EXAMPLE

The free group given by X_0 consists of strings made up of elements of X_0 and their formal inverses. For example, $x_1x_3^{-1}x_0x_2x_0^{-1}$ is in the free group; however, $x_2x_2^{-1}$ simplifies to e.

We have shown that if $\{A_\alpha\}$ is a family of $\mathcal{V}(S)$ algebras, then $A = \Pi A_\alpha \in \mathcal{V}(S)$. If $p_\alpha : A \to A_\alpha$ is defined by $p_\alpha(a) = a_\alpha$, recall that the following holds [see Section 3]:

Whenever $f_\alpha : B \to A_\alpha$ is a homomorphism for each α, there is a unique homomorphism $f : B \to \Pi A_\alpha$ such that $f_\alpha = p_\alpha f$ for all α.

The coproduct comes from reversing the arrows. It can be seen to combine algebras together, and can always be found due to the existence of free algebras.

DEFINITION

If $\{A_\alpha\}$ is a batch of $\mathcal{V}(S)$ algebras, a ***coproduct*** *[or* ***sum****] of the A_α's is defined to be an algebra $A \in \mathcal{V}(S)$ along with homomorphisms $i_\alpha : A_\alpha \to A$ such that whenever $B \in \mathcal{V}(S)$ and $f_\alpha : A_\alpha \to B$ for each α, there is a unique homomorphism $f : A \to B$ such that $fi_\alpha = f_\alpha$ for all α. The i_α are called* ***injection maps****.*

EXAMPLES

1. Coproducts in the variety of sets are disjoint unions, because any two maps $A \to C, B \to C$ combine to a unique map $A \uplus B \to C$.
2. If the $\{A_\alpha\}$ are R-modules, then their coproduct [normally called their direct sum] is the set ΣA_α of $a \in \Pi A_\alpha$ such that $a_\alpha \neq 0$ for finitely many α's. The injection map $i_\alpha : A_\alpha \to \Sigma A_\alpha$ is defined by $i_\alpha(a)_\alpha = a$, $i_\alpha(a)_\beta = 0$ when $\beta \neq \alpha$.
3. Exercise 14 shows that a coproduct of groups is a free product. If the G_α are groups, every nonidentity element of $\coprod G_\alpha$ can be written uniquely in the form $x_1x_2\ldots x_n$ where each $x_i \in G_\alpha$ for some α, $x_i \neq e$ and x_i, x_{i+1} are never in the same operand group.

LEMMA 1.23 *Every $\mathcal{V}(S)$ algebra is a homomorphic image of a free $\mathcal{V}(S)$ algebra.*

Proof of Lemma 1.23. If A is a $\mathcal{V}(S)$ algebra, the identity map $A \to A$ [where the domain is regarded as a set] extends to a homomorphism $f : F_S(\Omega, A) \to A$ satisfying $f(\overline{x}) = x$ for all $x \in A$. Each $a \in A$ is equal to $f(\overline{a})$, so f is surjective. Hence, A is a homomorphic image of $F_S(\Omega, A)$. ■

THEOREM 1.24 *Any $\mathcal{V}(S)$ algebras have a coproduct in $\mathcal{V}(S)$, which is unique up to isomorphism.*

The proof yields a legitimate recipe for finding coproducts in $\mathcal{V}(S)$. However, this recipe is deferred to Section 2.4, because it [sadly] makes this section too long.

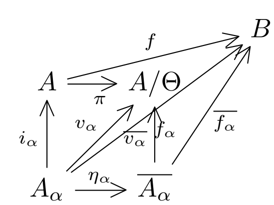
Proof of Theorem 1.24. We claim that $\mathcal{V}(S)$ algebras which possess coproducts include all free algebras and are closed under homomorphic images. Then since every algebra is a homomorphic image of a free algebra by Lemma 1.23, it will follow that all algebras have coproducts.

First suppose A is a coproduct of the A_α given by homomorphisms $i_\alpha : A_\alpha \to A$ and for each α, $\eta_\alpha : A_\alpha \to \overline{A_\alpha}$ is a surjective homomorphism. Now let $\Theta_\alpha = \ker \eta_\alpha$, $i_\alpha\Theta_\alpha = \{(i_\alpha(a), i_\alpha(b)) \mid (a, b) \in \Theta_\alpha\}$ and Θ the congruence relation on A generated by $\bigcup i_\alpha\Theta_\alpha$. [This constitutes our first trick!] We claim that A/Θ is a coproduct of the $\overline{A_\alpha}$.

Let π be the canonical epimorphism $A \to A/\Theta$ and $v_\alpha = \pi i_\alpha : A_\alpha \to A/\Theta$. We claim that $\Theta_\alpha \subseteq \ker v_\alpha$. To show this, suppose $(a, b) \in \Theta_\alpha$. Then $(i_\alpha(a), i_\alpha(b)) \in i_\alpha\Theta_\alpha \subseteq \Theta$. Hence, $\pi i_\alpha(a) = \pi i_\alpha(b)$ by definition of π, and $v_\alpha(a) = v_\alpha(b)$, which means that $(a, b) \in \ker v_\alpha$. Therefore, there is a unique homomorphism $\overline{v_\alpha} : \overline{A_\alpha} \to A/\Theta$ such that $v_\alpha = \overline{v_\alpha}\eta_\alpha$.

So we have a homomorphism from each $\overline{A_\alpha}$ to A/Θ. Now suppose $B \in \mathcal{V}(S)$ and $\overline{f_\alpha} : \overline{A_\alpha} \to B$ are homomorphisms. Let $f_\alpha = \overline{f_\alpha}\eta_\alpha : A_\alpha \to B$. Then since A is a coproduct of the A_α's, there is a unique homomorphism $f : A \to B$ such that $f_\alpha = f i_\alpha$ for all α.

The statements $v_\alpha = \pi i_\alpha, v_\alpha = \overline{v_\alpha}\eta_\alpha, f_\alpha = \overline{f_\alpha}\eta_\alpha, f_\alpha = f i_\alpha$ are organized in the following commutative diagram.



We claim that $\Theta \subseteq \ker f$: since Θ is *generated* by $\bigcup i_\alpha\Theta_\alpha$, we need only show that $i_\alpha\Theta_\alpha \subseteq f$ for every α to prove our claim. Whenever $(a', b') \in i_\alpha\Theta_\alpha$, there exists $(a, b) \in \Theta_\alpha$ such that $a' = i_\alpha(a)$ and $b' = i_\alpha(b)$. Furthermore,

$\eta_\alpha(a) = \eta_\alpha(b)$ so that $f(a') = fi_\alpha(a) = f_\alpha(a) = \overline{f_\alpha}\eta_\alpha(a) = \overline{f_\alpha}\eta_\alpha(b) = f_\alpha(b) = fi_\alpha(b) = f(b')$, and $(a', b') \in \ker f$. Therefore, $\Theta \subseteq \ker f$.

Consequently, f injectifies to a homomorphism $\overline{f} : A/\Theta \to B$ such that $f = \overline{f}\pi$, where π is the canonical epimorphism $A \to A/\Theta$. We also have $\overline{f}\overline{v_\alpha} = \overline{f_\alpha}$ because $\overline{f}\overline{v_\alpha}\eta_\alpha = \overline{f}v_\alpha = \overline{f}\pi i_\alpha = fi_\alpha = f_\alpha = \overline{f_\alpha}\eta_\alpha$, and η_α can be cancelled off the right due to surjectivity.

To show that $\overline{f}$ is unique, suppose $\overline{f}' : A/\Theta \to B$ also satisfies $\overline{f}'\overline{v_\alpha} = \overline{f_\alpha}$. Then $f' = \overline{f}'\pi$ satisfies $f'i_\alpha = \overline{f}'\pi i_\alpha = \overline{f}'v_\alpha = \overline{f}'\overline{v_\alpha}\eta_\alpha = \overline{f_\alpha}\eta_\alpha = f_\alpha$. Since f is the *unique* homomorphism $A \to B$ such that $fi_\alpha = f_\alpha$, we must have $f = f'$. Therefore, $\overline{f}\pi = \overline{f}'\pi$, hence $\overline{f} = \overline{f}'$ since π is surjective. Therefore, $\overline{f}$ is unique, and A/Θ is a coproduct of the $\overline{A_\alpha}$'s.

Now we show that free $\mathcal{V}(S)$ algebras have a coproduct. Let X_α be sets and $A_\alpha = F_S(\Omega, X_\alpha)$. Now let $X = \biguplus X_\alpha$ and we show that $A = F_S(\Omega, X)$ is a coproduct of the A_α's, with $i_\alpha : A_\alpha \to A$ mapping each element of X_α to the corresponding element of X.

Suppose B is an Ω-algebra and $f_\alpha : A_\alpha \to B$ for each α. Define $\overline{f} : X \to B$ mapping each $x \in X_\alpha$ to $f_\alpha(x)$. Then since $A = F_S(\Omega, X)$, there is a unique $f : A \to B$ such that $\overline{f} = f|X$. $f_\alpha = fi_\alpha$ for each α follows from $f_\alpha|X_\alpha = fi_\alpha|X_\alpha$, and $f|X$ is uniquely determined by this property, making f unique. This concludes the proof of the coproduct's existence.

The uniqueness of the coproduct is similar to Theorem 1.19 and is left to the reader. ■

EXERCISES

1. If $\Omega(0) = \{e\}$, $\Omega(1) = \{i\}$, $\Omega(2) = \{p\}$ and

   $$S = \{((p(px_0x_1)x_2), (px_0(px_1x_2))), ((p(e)x_0), x_0), ((p(ix_0)x_0), (e))\}$$

   then $\mathcal{V}(S)$ is the class of groups. [*Hint*: Exercise 4(a) of Section 1.]

2. Suppose $\Omega(0) = \{0, 1\}$, $\Omega(1) = \{n\}$, $\Omega(2) = \{s, p\}$ and S consists of the following pairs:

$((s(sx_0x_1)x_2), (sx_0(sx_1x_2)))$	$((p(px_0x_1)x_2), (px_0(px_1x_2)))$
$((sx_0x_1), (sx_1x_0))$	$((px_0(sx_1x_2)), (s(px_0x_1)(px_0x_2)))$
$((s(0)x_0), x_0)$	$((p(sx_0x_1)x_2), (s(px_0x_2)(px_1x_2)))$
$((sx_0(nx_0)), (0))$	$((p(1)x_0), x_0)$
	$((px_0(1)), x_0)$

   (a) Rewrite the operators and identities so they are easier to read.

   (b) Convince yourself that $\mathcal{V}(S)$ is the class of rings.

3. Express the class of rings with involution as a variety.

4. There are no axioms for the pointed set — it's just a set with a nullary operator. Does this prevent the pointed sets from being a variety?

5. The free monoid given by a set X consists of the strings made up of elements of X, including the empty string. For example, if $X = X_0$, one of the elements is $x_1x_4x_2x_2x_5x_1$.

6. (a) The commutative monoids form a variety.

   (b) Describe the free commutative monoid given by a set.

7. Every element of the free ring given by a set X is a formal sum of strings made up of elements of X and their formal negatives. For example, $x_1x_3 - x_2 + x_4x_6x_4$ is in the free ring given by X_0; and $(x_1+x_4)(x_2x_3+x_2)$ can be changed to $x_1x_2x_3 + x_4x_2x_3 + x_1x_2 + x_4x_2$ so it doesn't have parentheses.

8. If M is a fixed monoid, the free M-action given by a set X is $M \times X$ given by $m(a, x) = (ma, x)$ for $m, a \in M, x \in X$, and $i : X \to M \times X$ sending $x \to (1, x)$.

9. Let $X' = \{x_1, x_2, \ldots x_n\}$ and $v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_k$ be expressions in X'. Then

$$A = \langle x_1, x_2, \ldots x_n \mid v_1 = w_1, v_2 = w_2, \ldots, v_k = w_k \rangle$$

   is defined to be the result of taking the free $\mathcal{V}(S)$ algebra given by X', and then dividing out the congruence relation generated by the (v_j, w_j)'s. [This is usually done in the variety of groups.] If B is a $\mathcal{V}(S)$ algebra, show that a map $f : X' \to B$ extends to a homomorphism $A \to B$ if and only if substituting each x_i for $f(x_i)$ in any statement $v_j = w_j$ yields a true statement in B.

10. Let $I_S(\Omega) = F_S(\Omega, \emptyset)$. $I_S(\Omega)$ is called the **initial algebra for the variety** $\mathcal{V}(S)$.

    (a) $I_S(\Omega)$ is nonempty if and only if Ω contains a nullary operator.

    (b) If $\mathcal{V}(S)$ is the class of rings, $I_S(\Omega) \cong \mathbb{Z}$. [*Hint*: Exercise 7.]

    (c) For each $A \in \mathcal{V}(S)$, there is exactly one homomorphism $I_S(\Omega) \to A$, and its image is the smallest subalgebra of A.

    (d) A $\mathcal{V}(S)$ algebra is a homomorphic image of $I_S(\Omega)$ if and only if it has no subalgebra except itself.

11. Assume $A \coprod B$ denotes a coproduct of A and B in $\mathcal{V}(S)$.

    (a) If $A \cong C$ and $B \cong D$, then $A \coprod C \cong B \coprod D$

    (b) $(A \coprod B) \coprod C \cong A \coprod (B \coprod C)$

    (c) $A \coprod B \cong B \coprod A$

    (d) $I_S(\Omega) \coprod A \cong A$

12. If $S \subseteq T \subseteq F(\Omega, X_0)$, every Ω-algebra in $\mathcal{V}(T)$ is in $\mathcal{V}(S)$. [$\mathcal{V}(T)$ is said to be a **subvariety** of $\mathcal{V}(S)$ in this case.]

13. Suppose $\mathcal{V}(S)$ is a variety in which $I_S(\Omega) \cong T(\Omega)$. Then every $\mathcal{V}(S)$ algebra has a unique one-element subalgebra. Furthermore, for all $A, B \in \mathcal{V}(S)$, there exists a homomorphism $A \to B$. [If the initial algebra is isomorphic to the terminal algebra, it can be called a **zero algebra**.]

14. A coproduct $\coprod A_\alpha$ of $\mathcal{V}(S)$ algebras is said to be a **free product** if every $i_\alpha : A_\alpha \to \coprod A_\alpha$ is injective. If a homomorphism $A_\alpha \to A_\beta$ exists for all α, β, then $\coprod A_\alpha$ is a free product. [*Hint*: For each α, let $f_\beta : A_\beta \to A_\alpha$ be any homomorphisms, subject to the condition that $f_\alpha = 1_{A_\alpha}$. There is a homomorphism $f : \coprod A_\alpha \to A_\alpha$ such that $fi_\beta = f_\beta$ for all β. Use this to show that i_α is injective.]

15. (a) If $\mathcal{V}(S)$ is a variety in which all operators are nullary, when is a $\mathcal{V}(S)$ algebra free?

    (b) If $\mathcal{V}(S)$ is a variety in which all operators are unary, show that $\mathcal{V}(S)$ is the variety of M-actions for some fixed monoid M. Conclude that coproducts in $\mathcal{V}(S)$ are disjoint unions, and subalgebras of a $\mathcal{V}(S)$ algebra include the empty set and are closed under unions.