1.8 - Ω-expressions and Free Ω-algebras

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Have you ever attempted to treat a job fairly, and not regard any rules? If so, you'd realize how tough it is. Well, that's one of the many important concepts when it comes to universal algebra.

The free algebra will bring light to many of the future lessons. It takes symbols and shells them with operators, disregarding what they could possibly mean. Having done so, the symbols can map to any elements of a particular Ω-algebra, and this gives rise to a homomorphism.

First, we define Ω-expressions on a set X. We also add a notion of *length* so the expressions stay finite.

DEFINITION

*If X is a set and Ω a universal algebra template, an **Ω-expression in** X is recursively defined as follows.*

1. If α is an element of X, then α is an Ω-expression in X with length 1.

2. If $\omega \in \Omega(n)$ and $\alpha_1, \alpha_2, \dots \alpha_n$ are Ω-expressions with lengths $k_1, k_2, \dots k_n$ respectively, the expression $(\omega\alpha_1\alpha_2 \dots \alpha_n)$ is an Ω-expression whose length is $1 + k_1 + k_2 + \dots + k_n$.

In particular, if $\omega \in \Omega(0)$, (ω) is an Ω-expression with length 1.

For example, if s and p are the sum and product in a ring, $(px(syz))$ is an Ω-expression in $\{x, y, z\}$ of length 5; whereas $(s(pxy)(pxz))$ — which is supposed to be the same thing — is an Ω-expression of length 7.

We let $F(\Omega, X)$ be the set of Ω-expressions in X, and define its Ω-algebra structure as follows: if $\omega \in \Omega(n)$ and $\alpha_1, \alpha_2, \dots \alpha_n \in F(\Omega, X)$, then $(\omega\alpha_1\alpha_2 \dots \alpha_n)$ is the element of $F(\Omega, X)$ given by part 2 of the definition. We then define $i : X \to F(\Omega, X)$ mapping each element of X to itself as an Ω-expression [part 1 of the definition]. $F(\Omega, X)$ is called the **free Ω-algebra** given by X.

The first important thing to realize is this: let $A = \langle i(X) \rangle \subseteq F(\Omega, X)$. We show that $A = F(\Omega, X)$. For each $a \in F(\Omega, X)$, we induct on the length of a to show that $a \in A$. If a has length 1, it is either an element of X or a nullary operator (ω), and in either case $a \in A$, because A is generated by the set $i(X)$ of elements of X seen as Ω-expressions, and it contains all nullary operators due to being a subalgebra. Now suppose a has length $n \geq 2$ and every expression with length $< n$ is in A. Then $a = (\omega a_1 a_2 \dots a_k)$ with $\omega \in \Omega(k)$. Each a_i has length less than that of a, hence is in A by the inductive hypothesis. Since A is a subalgebra, $a \in A$ follows. Hence,

The free algebra $F(\Omega, X)$ is generated by the set $i(X)$ of symbols in X.

Now suppose that A is an Ω-algebra and $f : X \to A$ a set map. Define $f_1 : F(\Omega, X) \to A$ by assigning expressions in increasing order of length: map $x \in X$ to $f(x)$, and $(\omega a_1 a_2 \dots a_n)$ to $(\omega f_1(a_1) f_1(a_2) \dots f_1(a_n))$ [Once you get to $(\omega a_1 a_2 \ldots a_n)$, the $f_1(a_i)$'s already exist]. Then $f_1(\omega a_1 a_2 \ldots a_n) = (\omega f_1(a_1) f_1(a_2) \ldots f_1(a_n))$ is immediate from the definition, and $f_1 i = f$, because for each $x \in X$, $f_1 i(x)$ is f_1's assignment of the expression x, which is $f(x)$.

It is also seen that f_1 is uniquely determined by being a homomorphism $F(\Omega, X) \to A$ satisfying $f_1 i = f$, since $i(X)$ generates $F(\Omega, X)$; hence if $f_1' : F(\Omega, X) \to A$ is also a homomorphism satisfying $f_1' i = f$, then $f_1(x) = f_1'(x)$ for all $x \in i(X)$, hence $f_1 = f_1'$ by Exercise 10(b) of Section 3. This illustrates the following definition.

DEFINITION

Let $\mathcal{C}$ be a class of Ω-algebras. If X is a set, $F \in \mathcal{C}$ and $i : X \to F$ is a set map, (F, i) is said to constitute ***a free algebra for $\mathcal{C}$ given by*** *X provided that whenever $A \in \mathcal{C}$ and $f : X \to A$ is a set map, there is a unique homomorphism $f_1 : F \to A$ such that $f_1 i = f$.*

In particular, we have just seen that $F(\Omega, X)$ along with i is a free algebra for all Ω-algebras given by X. It is in fact the only one, as the following theorem shows.

THEOREM 1.19 *Let $\mathcal{C}$ be a class of Ω-algebras. Assume (F, i) constitutes a free algebra given by a set X, and (F', i') constitutes a free algebra given by a set X'. If $|X| = |X'|$ then $F \cong F'$.*

Basically, if there's a free algebra given by a set with a given cardinality, it is unique up to isomorphism.

Proof of Theorem 1.19. Let $\sigma : X \to X'$ be the bijection which is hypothesized to exist. Consider the map $i'\sigma : X \to F'$; since F is free given by X, there is a homomorphism $f : F \to F'$ satisfying $fi = i'\sigma$. Now reverse the roles and consider $i\sigma^{-1} : X' \to F$: since F' is free given by X', there is a homomorphism $f' : F' \to F$ satisfying $f'i' = i\sigma^{-1}$. The map $f'f : F \to F$ satisfies $f'fi = f'i'\sigma = i\sigma^{-1}\sigma = i1_X = i$. Since F is free, however, 1_F is the *unique* homomorphism $F \to F$ satisfying $1_F i = i$. Therefore, $f'f = 1_F$ by uniqueness. By the same argument, $ff' = 1_{F'}$. Hence, f and f' are isomorphisms which are inverses of each other, and $F \cong F'$. ■

EXERCISES

1. Let $\mathcal{C}$ be a class of Ω-algebras, X a set, and $F \in \mathcal{C}$ with $i : X \to F$ a free algebra for $\mathcal{C}$ given by X.

   (a) If any subalgebra of an algebra in $\mathcal{C}$ is in $\mathcal{C}$, then $i(X) \subseteq F$ generates F. [*Hint*: Let $A = \langle i(X) \rangle \subseteq F$. The map $X \to A$ sending $x \to i(x)$ extends to a homomorphism $\lambda : F \to A$ sending $i(x) \to i(x)$, since F is free. But there's also the canonical monomorphism $\iota : A \hookrightarrow F$. What can you say about $\iota\lambda : F \to F$?]

   (b) If $\mathcal{C}$ contains an algebra with at least two elements, then i is injective.

2. Let $I(\Omega) = F(\Omega, \emptyset)$.

   (a) For each Ω-algebra A, there is exactly one homomorphism $I(\Omega) \to A$. [$I(\Omega)$ is called the **initial Ω-algebra**.]

   (b) $I(\Omega) \neq \emptyset$ if and only if $\Omega(0) \neq \emptyset$.

3. The relation in $F(\Omega, X)$ of having the same length is a congruence relation.

4. If $X' \subseteq X$, then the subalgebra $\langle i(X') \rangle$ of $F(\Omega, X)$ is isomorphic to $F(\Omega, X')$.

5. Describe the congruence relation on $F(\Omega, X)$ generated by $\{(i(x), i(y)) \mid x, y \in X\}$. If $X \neq \emptyset$, what is the quotient algebra?

6. Let Φ be an equivalence relation on the set X and Φ_1 the congruence relation on $F(\Omega, X)$ generated by $\{(i(x), i(y)) \mid x\Phi y \text{ in } X\}$. Then $F(\Omega, X)/\Phi_1 \cong F(\Omega, X/\Phi)$. [*Hint*: Consider the map $X \to F(\Omega, X/\Phi)$ sending each $x \in X$ to $\overline{x}_\Phi$ as an expression in the free algebra. Then extend its domain, and injectify.]

7. In the class of pointed sets, the free algebra given by a set X is the set $X \uplus \{x_0\}$ with base point x_0. Conclude that every pointed set is actually free.

8. In the class of all sets, what's the free algebra given by a set X?

9. If X is a set and $f : X \to A$ a set map, then the resulting extension $F(\Omega, X) \to A$ has image $\langle f(X) \rangle \subseteq A$.

10. An element $a \in A$ is said to be **derivable** from $X \subseteq A$ provided that there exists an Ω-expression in X which evaluates to a in A. Show that $\langle X \rangle$ is the set of elements of A derivable from X. [*Hint*: Exercise 9.]