1.7 - The Ultraproduct

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

This section is not a prerequisite of any other and may be skipped if desired.

Isn't it saddening that some objects just fail the product? For example, if F and G are fields, the ring product $F \times G$ is *not* a field: $(1, 0)$ is a nonzero element of $F \times G$ which doesn't have an inverse. Thing is, the multiplicative inverse isn't really a unary operator, it's a *partial* unary operator because 0^{-1} is not defined.

That's just an ordinary product all right, it's not meant to preserve much. But what kind of a product would preserve *all* the logic there is? The ultrafilter is what brightens our day at this very moment. Let's just start out with the filter, which was mentioned in Section 4.

A filter in a Boolean algebra B is a nonempty subset F satisfying the following two properties:

(1) Whenever $x \in F$ and $y \in F$, $x \wedge y \in F$;

(2) Whenever $x \in F$ and $x \leq y$ then $y \in F$.

Theoretically, every filter F contains $1 \in B$, but if $0 \in F$ then $F = B$. This bears a similarity to ideals in a ring: every ideal contains 0, but if 1 is in there, it's the whole ring.

LEMMA 1.16 *Let F be a proper filter in a Boolean algebra B. Then F is maximal in the poset of proper filters in B if and only if whenever $a \in B$, either $a \in F$ or $a' \in F$.*

A proper filter satisfying either of the equivalent conditions in Lemma 1.16 is called an **ultrafilter**. Note, by the way, that we can't have *both* a and a' in F, because that would cause $a \wedge a' = 0 \in F$ and $F = B$. Hence an ultrafilter contains a if and only if it doesn't have a'.

Proof of Lemma 1.16. Suppose that F is maximal in the lattice of proper filters and $a \in B$. We want to show that $a \in F$ or $a' \in F$. If $a \in F$, there is nothing to prove. If $a \notin F$, define

$$G = \{x \in B \mid a \wedge u \leq x \text{ for some } u \in F\}.$$

Then G is seen to be a filter in B: $a \wedge u \leq 1$ for all $u \in F$, so $1 \in G$ and G is nonempty. If $x, y \in G$, then there exist $u, v \in F$ such that $a \wedge u \leq x$ and $a \wedge v \leq y$. Furthermore, $u \wedge v \in F$ and $a \wedge u \wedge v \leq a \wedge u \leq x$ and $a \wedge u \wedge v \leq a \wedge v \leq y$, so that $a \wedge u \wedge v \leq x \wedge y$. This means $x \wedge y \in G$. If $x \in G$ and $x \leq y$, then $a \wedge u \leq x$ for some $u \in F$; since $a \wedge u \leq x \leq y$, we have $a \wedge u \leq y$ and $y \in G$. Therefore, G is a filter in B.

Furthermore, $F \subseteq G$, because if $x \in F$, then $a \wedge x \leq x$ so $a \wedge u \leq x$ for some $u \in F$, thus $x \in G$. Also, $a \in G$ because $1 \in F$ and $a \wedge 1 \leq a$, but we are given $a \notin F$. Therefore, $F \neq G$. Since F is maximal we must have $G = B$.

In particular, $0 \in G$, so that $a \wedge u \leq 0$ for some $u \in F$. Since 0 is smallest in B, this basically says that $a \wedge u = 0$; hence, $u \wedge a' = (u \wedge a') \vee 0 = (u \wedge a') \vee (u \wedge a) = u \wedge (a' \vee a) = u \wedge 1 = u$ and $u \leq a'$. Since $u \in F$, $a' \in F$ follows, completing the proof of this implication.

Conversely, if $a \in F$ or $a' \in F$ for all $a \in B$, suppose G is a filter in B with $F \subsetneqq G$. Then there exists $a \in G$ such that $a \notin F$. Since $a \notin F$, $a' \in F$ by hypothesis, hence $a' \in G$. Therefore $a \wedge a' = 0 \in G$ and $G = B$. Thus, F is maximal. ■

For the remainder of this section, we deal with filters in the power set $\mathcal{P}(I)$ [the subsets of I under inclusion] where I is a set of indices. If $\{A_\alpha\}$ is an indexed set of Ω-algebras and F is a filter in $\mathcal{P}(I)$, let Φ be the relation on the product ΠA_α given by

$$a \Phi b \text{ if } \{\alpha \in I \mid a_\alpha = b_\alpha\} \in F.$$

Stated otherwise, a and b are congruent if F "has the set of their shared components." We show that Φ is truly a congruence relation on ΠA_α. For $a \in A$, $\{\alpha \in I \mid a_\alpha = a_\alpha\} = I \in F$, hence, $a \Phi a$ and Φ is reflexive. If $a \Phi b$, then $\{\alpha \in I \mid b_\alpha = a_\alpha\} = \{\alpha \in I \mid a_\alpha = b_\alpha\} \in F$, so $b \Phi a$ easily follows. Now suppose $a \Phi b$ and $b \Phi c$. Then $A_1 = \{\alpha \in I \mid a_\alpha = b_\alpha\}$ and $A_2 = \{\alpha \in I \mid b_\alpha = c_\alpha\}$ are in F. To show that Φ is transitive, we need to show that $A_3 = \{\alpha \in I \mid a_\alpha = c_\alpha\}$ is in F, so that $a \Phi c$. Well, if $\alpha \in A_1 \cap A_2$, then $a_\alpha = b_\alpha = c_\alpha$ and $\alpha \in A_3$. Therefore, $A_1 \cap A_2 \subseteq A_3$. Yet, $A_1 \cap A_2 \in F$ and $A_3 \in F$ follow since F is a filter. Consequently, Φ is an equivalence relation.

Now suppose $\omega \in \Omega(n)$, $a^1, a^2, \ldots a^n, b^1, b^2, \ldots b^n \in \Pi A_\alpha$ and $a^i \Phi b^i$ for every i. Then for each i, form the set $A_i = \{\alpha \in I \mid a^i_\alpha = b^i_\alpha\}$. We are given that F contains all the A_i's, and need to show that F contains $A = \{\alpha \in I \mid (\omega a^1 a^2 \ldots a^n)_\alpha = (\omega b^1 b^2 \ldots b^n)_\alpha\}$. It is seen that $A_1 \cap A_2 \cap \cdots \cap A_n \subseteq A$, by reasoning similar to the last paragraph, from which $A \in F$ follows. Therefore, Φ is indeed a congruence relation on ΠA_α.

The special moment comes from the ultrafilter.

DEFINITION

Let $\{A_\alpha\}$ *be an indexed collection of* Ω*-algebras with indices in* I*, and* U *an ultrafilter in* $\mathcal{P}(I)$*. If* Φ *is defined as before [*$a \Phi b \iff \{\alpha \in I \mid a_\alpha = b_\alpha\} \in U$*], the quotient algebra* $(\Pi A_\alpha)/\Phi$ *is called an* ***ultraproduct*** *of the* A_α*'s.*

Fields fail the product, as previously seen. But they don't fail the ultraproduct — and almost nothing fails this. Let's practice a proof.

THEOREM 1.17 *An ultraproduct of fields is a field.*

Proof of Theorem 1.17. Let $\{A_\alpha\}$ be an indexed collection of fields and $(\Pi A_\alpha)/\Phi$ the ultraproduct of the A_α's involving ultrafilter U. It is clear that $(\Pi A_\alpha)/\Phi$ is a commutative ring, because commutative rings are closed under homomorphic images and products.

Now suppose $\overline{a} \neq \overline{0}$ in $(\Pi A_\alpha)/\Phi$. We show that $\overline{a}\overline{b} = \overline{1}$ for some $\overline{b}$. $\overline{a} \neq \overline{0}$ implies that $\{\alpha \in I \mid a_\alpha = 0\} \notin U$, so its complement, $N = \{\alpha \in I \mid a_\alpha \neq 0\}$ *is* in U. Consider b given by $b_\alpha = a_\alpha^{-1}$ if $a_\alpha \neq 0$, and 0 if $a_\alpha = 0$. Then $(ab)_\alpha = a_\alpha b_\alpha$ is 1 if $a_\alpha \neq 0$ and 0 if $a_\alpha = 0$. We claim that $\overline{a}\overline{b} = \overline{ab} = \overline{1}$; to show this, we see that $\{\alpha \in I \mid (ab)_\alpha = 1\} = \{\alpha \in I \mid a_\alpha \neq 0\} = N \in U$. Therefore, $\overline{a}\overline{b}$ is indeed $\overline{1}$, and $(\Pi A_\alpha)/\Phi$ is a field. ■

An ultraproduct of integral domains is an integral domain by the same token [see Exercise 1]. This can be generalized.

What are examples of filters in $\mathcal{P}(I)$? Well, a lot can be said about them.

EXAMPLES

1. A subset S of I is said to be **cofinite** in I if $I - S$ is finite. It is easily seen that the set $\mathcal{F}$ of cofinite subsets of I is a filter in $\mathcal{P}(I)$. It is not an ultrafilter; for example, if $I = \mathbb{Z}$, $\mathcal{F}$ contains neither the set of even integers, nor its complement, the set of odd integers. $\mathcal{F}$ is called the **Fréchet filter**.

2. If $\sigma \in I$, then the set of subsets of I containing σ is an ultrafilter in $\mathcal{P}(I)$. It is called **the principal ultrafilter given by** σ.

So yeah, a basic example of an ultrafilter is there. However, if U is the principal ultrafilter given by $\sigma \in I$, the ultraproduct $(\Pi A_\alpha)/\Phi$ is either empty or isomorphic to the operand A_σ [Exercise 4]. So of course it satisfies everything all the operands satisfy. So principal ultrafilters don't really produce much. Now, does there exist a nonprincipal ultrafilter? Well, the Fréchet filter $\mathcal{F}$ allows us to answer this.

THEOREM 1.18 *An ultrafilter U in $\mathcal{P}(I)$ is nonprincipal if and only if $\mathcal{F} \subseteq U$.*

Proof of Theorem 1.18. Suppose U is the principal ultrafilter given by $\sigma \in I$. Then the set $I - \{\sigma\}$ is in $\mathcal{F}$ but not in U, hence $\mathcal{F}$ isn't contained in U. Thus if $\mathcal{F} \subseteq U$ then U is nonprincipal [we just proved the contrapositive]. Conversely, suppose $\mathcal{F}$ isn't contained in U; we are to show that U is principal. In this case, there exists $A \in \mathcal{F}$ such that $A \notin U$. Since $A \notin U$ and U is an ultrafilter, the complement $I - A \in U$. Since $A \in \mathcal{F}$, $I - A$ is finite by definition. Hence U contains a finite set. Let n be the smallest positive integer such that U contains a set with n elements. If $n \geq 2$, say $\{\alpha_1, \dots \alpha_n\} \in U$, then $\{\alpha_1\} \notin U$ [otherwise U would contain a set with fewer than n elements]. Therefore, $I - \{\alpha_1\} \in U$ since U is an ultrafilter. Intersecting this set with $\{\alpha_1, \dots \alpha_n\}$ yields $\{\alpha_2, \dots \alpha_n\}$, which is hence a set in U with $n - 1$ elements. This contradicts the hypothesis that n is the smallest integer such that U has a set with n elements. Therefore $n = 1$, hence U contains $\{\sigma\}$ for some $\sigma \in I$, and is easily seen to be the principal ultrafilter given by σ. ■

Therefore, the only question that remains is whether there's an ultrafilter containing $\mathcal{F}$. If I is finite, $\mathcal{F} = \mathcal{P}(I)$ so this is impossible [in other words, every ultrafilter in $\mathcal{P}(I)$ is principal if I is finite!]. So suppose I is infinite. Then

$\mathcal{F}$ doesn't contain $\emptyset$ and is hence proper in $\mathcal{P}(I)$. If we assume the Axiom of Choice, we can use Zorn's Lemma [see Exercise 4 of Section 6] to show that an ultrafilter containing $\mathcal{F}$ [hence nonprincipal] exists.

Let S be the set of proper filters in $\mathcal{P}(I)$ containing $\mathcal{F}$. Assuming I is infinite, $\mathcal{F} \in S$ so that S is nonempty. If $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$ is a chain of filters containing $\mathcal{F}$, their union is also a filter containing $\mathcal{F}$, and is proper if all the F_i's are [because a filter's proper if and only if it doesn't contain $\emptyset$]. Therefore, every chain in S has an upper bound in S, so S has a maximal element by Zorn's Lemma. Lemma 1.16 shows that this is an ultrafilter in $\mathcal{P}(I)$. And it clearly contains $\mathcal{F}$.

The conclusion is, therefore, that $\mathcal{P}(I)$ has a nonprincipal ultrafilter if and only if I is infinite.

EXERCISES

1. An ultraproduct of integral domains is an integral domain.

2. A monoid M is said to be **cancellative** provided that whenever $a, b, c \in M$ and $ab = ac$ or $ba = ca$, then $b = c$.

   (a) If monoids M and N are cancellative, it so happens that $M \times N$ is cancellative.

   (b) Why doesn't this imply that the product of integral domains is an integral domain?

   (c) A submonoid of a cancellative monoid M is cancellative.

   (d) If M is cancellative and Φ is a congruence relation on M, show by example that M/Φ need not be cancellative.

3. Let U be an ultrafilter in a Boolean algebra B, and $a, b \in B$. Then:

   (a) $a \wedge b \in U$ if and only if $a \in U$ and $b \in U$,

   (b) $a \vee b \in U$ if and only if either a or b is in U.

4. Let $\sigma \in I$ and U the principal ultrafilter in $\mathcal{P}(I)$ given by σ. Assume the A_α's are nonempty Ω-algebras.

   (a) The kernel of the projection homomorphism $p_\sigma : \Pi A_\alpha \to A_\sigma$ is the congruence relation Φ on ΠA_α given by the U in the definition of the ultraproduct.

   (b) In conclusion, the ultraproduct $(\Pi A_\alpha)/\Phi$ is isomorphic to A_σ.

5. An ultrafilter U in $\mathcal{P}(I)$ is principal if and only if the intersection of a [possibly infinite] batch of sets in U is in U.