1.6 - Subdirect Products

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Only section 10 will use this material, and the theorems used will be stated in the section.

The subdirect product cuts down on congruence relations in the strongest way. To illustrate this, let G be a group and N_1, N_2, N_3 be normal subgroups of G such that $N_1 \cap N_2 \cap N_3 = \langle e \rangle$. That last statement shows that no information is lost in all of G/N_1, G/N_2 and G/N_3. But does that mean we can get G back from them somehow?

The answer is yes, but it's not all that algorithmic. Define $f : G \to G/N_1 \times G/N_2 \times G/N_3$ by $f(a) = (N_1a, N_2a, N_3a)$. f is clearly a group homomorphism. Now if $a \in \ker f$, then $f(a) = (N_1e, N_2e, N_3e)$, whence $a \in N_1$, $a \in N_2$ and $a \in N_3$. Therefore $a \in N_1 \cap N_2 \cap N_3 = \langle e \rangle$, and $a = e$, from which it follows that $\ker f = \langle e \rangle$ and f is injective. Hence, *G is isomorphic to a subgroup of $G/N_1 \times G/N_2 \times G/N_3$* [namely, the image of f]. If we can find all subgroups of the direct product, we can get G.

Now we generalize this to universal algebra, with an arbitrary — *possibly infinite* — batch of congruence relations. Let A be an Ω-algebra and $\{\Phi_\alpha\}$ a batch of congruence relations on A such that $\cap\Phi_\alpha = 1_A$. Then there is no information in A that gets lost in all of the A/Φ_α.

Define $f : A \to \Pi A/\Phi_\alpha$ by $f(a)_\alpha = \overline{a}_{\Phi_\alpha}$. Then for $\omega \in \Omega(n)$, $a^1, a^2, \dots a^n \in A$ and component index α,

$$f(\omega a^1 a^2 \dots a^n)_\alpha = \overline{(\omega a^1 a^2 \dots a^n)}_{\Phi_\alpha} = (\omega \overline{a^1}_{\Phi_\alpha} \overline{a^2}_{\Phi_\alpha} \dots \overline{a^n}_{\Phi_\alpha})$$

$$= (\omega f(a^1)_\alpha f(a^2)_\alpha \dots f(a^n)_\alpha) = (\omega f(a^1) f(a^2) \dots f(a^n))_\alpha$$

hence f is a homomorphism. Now suppose $f(a) = f(b)$. Then for every α, $f(a)_\alpha = f(b)_\alpha$, hence $\overline{a}_{\Phi_\alpha} = \overline{b}_{\Phi_\alpha}$ and $a\Phi_\alpha b$. This means (a, b) is in every Φ_α, hence $(a, b) \in \cap\Phi_\alpha = 1_A$ and $a = b$. Therefore, f is injective. Note that if $p_\alpha : \Pi A/\Phi_\alpha \to A/\Phi_\alpha$ is the projection homomorphism $a \to a_\alpha$, then $p_\alpha f$ is the canonical epimorphism $A \to A/\Phi_\alpha$, and is hence surjective. This motivates the following definition.

DEFINITION

*If $\{A_\alpha\}$ is a batch of Ω-algebras, a **subdirect product** of the A_α is an Ω-algebra A along with an injective homomorphism $f : A \to \Pi A_\alpha$ such that for each projection $p_\alpha : \Pi A_\alpha \to A_\alpha$, the map $p_\alpha f$ is surjective.*

You can think of a subdirect product of the A_α's as a subalgebra of the product, such that for each component index α, every element of A_α lies in index α of some element.

As we have seen, if $\cap\Phi_\alpha = 1_A$, A is a subdirect product of the A/Φ_α's. If we aren't given $\cap\Phi_\alpha = 1_A$, then what? This is answered in Exercise 1.

Recall that whenever p is a prime integer and $p = nm$ with n and m integers, then $p = \pm n$ or $p = \pm m$. Subdirect irreducibility is defined similarly:

DEFINITION

*An Ω-algebra A is said to be **subdirectly irreducible** provided that $|A| \geq 2$ and whenever A is a subdirect product of $\{A_\alpha\}$ given by $f : A \to \Pi A_\alpha$, there exists a component index α such that $p_\alpha f : A \to A_\alpha$ is an isomorphism.*

Stated otherwise, for each $a \in A_\alpha$, there is exactly one $t \in f(A)$ with $t_\alpha = a$. Hence, A is simply isomorphic to the operand A_α.

Is there an easier way to think about this? When A is a subdirect product of $\{A_\alpha\}$, each A_α is a homomorphic image of A, with the kernels intersecting to 1_A. This fails when you cannot intersect congruence relations of A and result in 1_A, unless one of the operands itself is 1_A.

THEOREM 1.15 *An Ω-algebra A with $|A| \geq 2$ is subdirectly irreducible if and only if the intersection of all nonidentity congruence relations on A is not 1_A.*

Proof of Theorem 1.15. If A is subdirectly irreducible, let $\{\Phi_\alpha\}$ be the set of nonidentity congruence relations on A. We want to show that $\cap\Phi_\alpha \neq 1_A$. If $\cap\Phi_\alpha = 1_A$, define $f : A \to \Pi A/\Phi_\alpha$ by $f(a)_\alpha = \overline{a}_{\Phi_\alpha}$ as before. We have already seen this to be a subdirect product. By subdirect irreducibility, $p_\alpha f : A \to A/\Phi_\alpha$ is an isomorphism for some α. This means that $\ker(p_\alpha f) = 1_A$. But $\ker(p_\alpha f) = \Phi_\alpha$, since $p_\alpha f$ is the canonical epimorphism. Hence $\Phi_\alpha = 1_A$, contrary to $\{\Phi_\alpha\}$ being the set of *nonidentity* congruence relations. Therefore, $\cap\Phi_\alpha \neq 1_A$. Conversely, suppose the intersection of all nonidentity relations on A is not 1_A, and $f : A \to \Pi A_\alpha$ gives a subdirect product. It is clear that the kernel of f, which is 1_A, is the intersection of the kernels of $p_\alpha f$ for every α. Since nonidentity congruence relations never intersect to 1_A, $\ker(p_\alpha f) = 1_A$ for some α. Hence $p_\alpha f : A \to A_\alpha$ is injective, but it is also surjective by definition of a subdirect product, so that it is an isomorphism. Hence, A is subdirectly irreducible. ■

Exercise 4 shows that an Ω-algebra with at least two elements is a subdirect product of subdirectly irreducible algebras.

EXERCISES

1. Let $\{\Phi_\alpha\}$ be any batch of congruence relations on A, and let $\Phi = \cap\Phi_\alpha$. A/Φ is a subdirect product of the A/Φ_α's. [*Hint*: Define $f : A \to \Pi A/\Phi_\alpha$ by $f(a)_\alpha = \overline{a}_{\Phi_\alpha}$ as before. Find the kernel of f and injectify.]

2. (a) Use Theorem 1.15 to show that if $n \geq 2$, the cyclic group $\mathbb{Z}_n$ is subdirectly irreducible if and only if $n = p^k$ with p prime.

   (b) The ring $\mathbb{Z}_n$ is also subdirectly irreducible if and only if $n = p^k$ with p prime.

3. If $\Phi_1 \subseteq \Phi_2 \subseteq \Phi_3 \subseteq \ldots$ is an ascending chain of congruence relations on A, the union $\cup\Phi_i$ is a congruence relation. [*Hint*: Exercise 3 of Section 2.]

4. Assume that there is a maximal element in every nonempty poset in which every chain subset has an upper bound. This is **Zorn's Lemma**, but the proof of this requires the Axiom of Choice.

   (a) If $a, b \in A$ with $a \neq b$, then there exists a congruence relation $\Phi_{a,b}$ on A with $(a, b) \notin \Phi_{a,b}$ such that whenever Φ is another congruence relation with $\Phi_{a,b} \subsetneqq \Phi$, $(a, b) \in \Phi$.

   (b) $A/\Phi_{a,b}$ is subdirectly irreducible. [*Hint*: Recall Exercise 6(c) of Section 5. What can be said about nonidentity congruence relations on $A/\Phi_{a,b}$? Use Theorem 1.15.]

   (c) Every Ω-algebra A with $|A| \geq 2$ is a subdirect product of subdirectly irreducible algebras. [*Hint*: Consider $A/\Phi_{a,b}$ for all $a \neq b$ in A.]