1.4 - Congruence Relations

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Congruence relations are the seed to quotient algebras. To understand them, we first need a notion of relations [as mathematical objects]. If A is a set, then a **relation** on A hereby refers to a subset of $A \times A$. If $\Phi \subseteq A \times A$ and $a, b \in A$, one can write $a\Phi b$ if $(a, b) \in \Phi$.

Recall that a relation Φ is an *equivalence relation* if for all $a, b, c \in A$:
$a\Phi a$ [reflexivity];
$a\Phi b \implies b\Phi a$ [symmetry];
$a\Phi b, b\Phi c \implies a\Phi c$ [transitivity].

In this case, if $\overline{a}$ is the equivalence class $\{b \in A \mid b\Phi a\}$, then $\overline{a} = \overline{b}$ if $a\Phi b$, otherwise $\overline{a} \cap \overline{b} = \emptyset$. A congruence relation is even stronger than an equivalence relation. If you apply an n-ary operator to equivalence classes, you should get one equivalence class without any dependence of the operand representatives. This motivates the following definition.

DEFINITION

A ***congruence relation*** *[or* ***congruence****] on an Ω-algebra A is an equivalence relation on A which is a subalgebra of $A \times A$.*

This definition may appear to be confusing. How does one view a relation Φ as a subalgebra of $A \times A$? Well, recall that it's constructed as a subset of $A \times A$. So the definition says that whenever $n \geq 0$, $\omega \in \Omega(n)$ and $(a_1, b_1), (a_2, b_2), \ldots (a_n, b_n) \in A \times A$, we have $(\omega(a_1, b_1)(a_2, b_2) \ldots (a_n, b_n)) = ((\omega a_1 a_2 \ldots a_n), (\omega b_1 b_2 \ldots b_n)) \in A \times A$. Stated otherwise, if $a_i \Phi b_i$ for $1 \leq i \leq n$, then $(\omega a_1 a_2 \ldots a_n)\Phi(\omega b_1 b_2 \ldots b_n)$.

Notice that if $n = 0$, the previous statement says $(\omega_A)\Phi(\omega_A)$. However, that is an immediate consequence of reflexivity, so nullary operators need not be regarded in a congruence relation. Stated otherwise,

An equivalence relation Φ on an Ω-algebra A is a congruence relation if and only if whenever $n \geq 1$, $\omega \in \Omega(n)$, $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n \in A$ and $a_i \Phi b_i$ for each $i = 1, 2, \ldots, n$, then $(\omega a_1 a_2 \ldots a_n)\Phi(\omega b_1 b_2 \ldots b_n)$.

The equivalence classes of a congruence relation are usually called **congruence classes**.

EXAMPLES

1. The identity relation/diagonal $1_A = \{(a, a) \mid a \in A\}$ is a congruence relation on A because if $a_i = b_i$ for all i, clearly $(\omega a_1 a_2 \ldots a_n) = (\omega b_1 b_2 \ldots b_n)$. The full relation $A \times A$ is also a congruence relation.

2. If Φ is a congruence relation on A and B is a subalgebra of A, then $\Phi \cap (B \times B)$ is a congruence relation on B, called the **restriction of Φ to B**.

3. If A and B are Ω-algebras, define a relation A^* by $(a_1, b_1)A^*(a_2, b_2)$ if $b_1 = b_2$. A^* is seen to be a congruence relation.

4. In general, let Φ be a congruence relation on B. Then define Φ^* by $(a_1, b_1)\Phi^*(a_2, b_2)$ if $b_1 \Phi b_2$. Φ^* is a congruence relation; the last example was a special case with $\Phi = 1_B$.

If you didn't learn about congruence relations in earlier abstract algebra lessons, that's probably because they're usually associated with special kinds of subsets. In universal algebra, however, they are relations at best, so I would recommend you read this section to get the hang of them.

EXAMPLES

1. If Φ is a congruence relation on a group G, then the congruence class $N = \{a \in G \mid a\Phi e\}$ is a normal subgroup of G. Conversely, if N is a normal subgroup, the relation $\{(a, b) \mid ab^{-1} \in N\}$ is a congruence relation. This is seen to be a one-to-one correspondence. So normal subgroups are used for congruence in a group, rather than congruence relations as previously defined.

2. Let R be a fixed ring and M a left R-module. If Φ is a congruence relation on M [ex. $a\Phi b$ implies $ka\Phi kb$ for all $k \in R$], then $N = \{a \in M \mid a\Phi 0\}$ is a submodule; and if N is a submodule then $\{(a, b) \mid a - b \in N\}$ is a congruence relation on M. Again, this is a one-to-one correspondence. So congruence relations in a module are identified with submodules.

3. If Φ is a congruence relation on a ring R, the set $I = \{a \in R \mid a\Phi 0\}$ is an ideal, and every ideal I results in a congruence relation $\{(a, b) \mid a - b \in I\}$. This is a one-to-one correspondence. Congruence relations in a ring are viewed as ideals.

4. Let Φ be a congruence relation on a Boolean algebra B. Then $F = \{a \in B \mid a\Phi 1\}$ is a filter in B. Conversely, whenever F is a filter in B, the relation $\{(a, b) \mid a \vee b' \in F \text{ and } a' \vee b \in F\}$ is the corresponding congruence. The Boolean algebra's congruence relation is thus a filter.

We recall that the subalgebras of A form a complete lattice under inclusion. The same is true for congruence relations. As subsets of $A \times A$, one would already know the notion of inclusion and intersection of relations: If Θ and Φ are congruence relations, then $\Theta \subseteq \Phi$ if and only if whenever $a\Theta b$ for $a, b \in A$, then $a\Phi b$. If $\{\Phi_\alpha\}$ are congruence relations, then $\cap\Phi_\alpha$ is the relation $a \sim b$ that $a\Phi_\alpha b$ for every α. According to the following theorem, $\cap\Phi_\alpha$ is indeed a congruence.

THEOREM 1.7 *If A is an Ω-algebra, then Con A — the set of congruence relations on A — is a complete lattice under inclusion.*

Proof of Theorem 1.7. Con A is clearly a poset under inclusion, with largest element $A \times A$ and smallest element 1_A. Now suppose $\{\Phi_\alpha\}$ are congruence relations on A; we claim that $\cap\Phi_\alpha$ is a congruence relation. In that case, we can apply Lemma 1.1 and conclude that Con A is a complete lattice.

Since every Φ_α is a subalgebra of $A \times A$, so is the intersection $\cap\Phi_\alpha$ by Exercise 2 of Section 2. We need only show that $\cap\Phi_\alpha$ is an equivalence relation on A. It is clear that for $a \in A$, (a, a) is in every Φ_α, hence in the intersection, so that $\cap\Phi_\alpha$ is reflexive. If $(a, b) \in \cap\Phi_\alpha$, then $(a, b) \in \Phi_\alpha$ for every α. Since each Φ_α is symmetric it contains (b, a), which is thus in $\cap\Phi_\alpha$. This proves symmetry. Finally, if $(a, b), (b, c) \in \cap\Phi_\alpha$, then each Φ_α contains (a, b) and (b, c), hence (a, c) by transitivity. Therefore, $(a, c) \in \cap\Phi_\alpha$, and $\cap\Phi_\alpha$ is transitive, hence an equivalence relation. ■

And now, very pertinent in many studies is a congruence relation generated by a set. As in the case of subalgebras, if $X \subseteq A \times A$, $[X]$ is the intersection of all congruence relations on A containing X. [At least one exists, namely $A \times A$.]

THEOREM 1.8 *Let A be an Ω-algebra and X be a subset of a $A \times A$. Then:*
(1) $[X]$ *is a congruence relation on A containing X.*
(2) *Whenever Φ is a congruence relation on A containing X, $[X] \subseteq \Phi$.*
(3) $[X]$ *is the only congruence relation on A with properties (1) and (2).*

Proof of Theorem 1.8. Copy the proof of Theorem 1.3, translating $\langle X \rangle$ to $[X]$, subalgebras of A to congruence relations on A, Exercise 2 of Section 2 to Theorem 1.7, and B to Φ. ■

If Θ and Φ are congruence relations on A, then $\Theta \cup \Phi$ need not be a congruence relation. However, by Theorem 1.7, Θ and Φ do have a least upper bound $\Theta \vee \Phi$.

Also, the correspondence between normal subgroups of a group and congruence relations actually preserves the lattice structure; if N and M are normal subgroups, $N \subseteq M$ if and only if congruence mod N is contained in congruence mod M. Same for ideals and rings, etc.

EXERCISES

From this point on, words like "show that" and "prove that" are omitted for simplification. If an exercise is in the form of a statement, you are supposed to prove it.

1. (a) If $f : A \to B$ is a homomorphism of Ω-algebras and $\Theta = \{(a, b) \in A \times A \mid f(a) = f(b)\}$, then Θ is a congruence relation on A.

   (b) If Φ is a congruence relation on B and $f : A \to B$ is a homomorphism, $\Theta = \{(a, b) \in A \times A \mid f(a)\Phi f(b)\}$ is a congruence relation on A.

2. If Φ is a congruence relation on A and B is a subalgebra of A, then B is said to be **Φ-invariant** provided that whenever $a \in B$ and $a\Phi b$, then $b \in B$.

   (a) If $\{A_\alpha\}$ is a family of Φ-invariant subalgebras of A, the intersection $\cap A_\alpha$ is Φ-invariant.

   (b) A is Φ-invariant for every congruence relation Φ on A.

   (c) Every subalgebra of A is 1_A-invariant.

3. (a) If R is a ring, I an ideal in R and S a subring of R, let Φ be the congruence relation associated with I. Then S is Φ-invariant if and only if $I \subseteq S$.

   (b) State an analogous result for groups.

4. If Φ is a congruence relation on A and B is a subalgebra of A, define $B\Phi = \{a \in A \mid a\Phi b \text{ for some } b \in B\}$.

   (a) $B\Phi$ is the smallest Φ-invariant subalgebra of A containing B.

   (b) B is Φ-invariant if and only if $B\Phi = B$.

   (c) If Φ_1 and Φ_2 are congruences on A, then $B(\Phi_1 \cap \Phi_2) \subseteq B\Phi_1 \cap B\Phi_2$.

   (d) Show by example that $B(\Phi_1 \cap \Phi_2) = B\Phi_1 \cap B\Phi_2$ may not hold.

   (e) If B_1 and B_2 are subalgebras of A, then $(B_1 \cap B_2)\Phi \subseteq B_1\Phi \cap B_2\Phi$.

   (f) Show by example that $(B_1 \cap B_2)\Phi = B_1\Phi \cap B_2\Phi$ may not hold.

5. (a) If Φ is a congruence relation on A and $\{\epsilon\}$ a one-element subalgebra of A, then $\overline{\epsilon} = \{a \in A \mid a\Phi\epsilon\}$ is a subalgebra of A. Conclude that if A has a one-element subalgebra, each congruence relation has a subalgebra of A as a congruence class.

   (b) Show by example that Φ need not be determined by $\overline{\epsilon}$.

6. (a) If G is a group, N a normal subgroup and K any subgroup, let Φ be the congruence relation associated with N. Then $K\Phi$ is the subgroup $NK = \{nk \mid n \in N, k \in K\}$.

   (b) $N \cap K$ is a normal subgroup of K, which corresponds to the congruence relation $\Phi \cap (K \times K)$ on K.

   (c) State an analogous result for rings.

7. If Θ and Φ are congruence relations on A, then $[\Theta \cup \Phi]$ is the least upper bound of Θ and Φ in the lattice of congruence relations.

8. An **ideal** in a Lie algebra L over a field F is a subalgebra I such that whenever $a \in I$ and $l \in L$, then $[a : l] \in I$ and $[l : a] \in I$. [Note that the statement $[l : a] \in I$ is superfluous because $[l : a] = -[a : l]$.] Show that there is a bijection between the ideals in L and the congruence relations on L, preserving the lattice structure.