1.3 - Homomorphisms and Isomorphisms

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

A homomorphism of Ω-algebras is a map that preserves structure. These maps are more important than any maps, since structure is mostly what we care for. To get a rigorous definition of a homomorphism, we must recall the structure of an Ω-algebra.

If A is an Ω-algebra, an n-ary operator corresponds to a map $\omega : A^n \to A$ mapping $a_1, a_2, \dots a_n$ to $(\omega a_1 a_2 \dots a_n)$. Now let $f : A \to B$ be a function. It is generous to say that f *preserves* ω if whenever it maps $a_i \in A$ to $b_i \in B$, it maps $(\omega a_1 a_2 \dots a_n)$ to $(\omega b_1 b_2 \dots b_n)$. But we want a quicker way to say this.

Notice that the element of B that f maps a_i to is denoted $f(a_i)$. Taking $b_i = f(a_i)$, we have $(\omega b_1 b_2 \dots b_n) = (\omega f(a_1) f(a_2) \dots f(a_n))$. So our rule is that f maps $(\omega a_1 a_2 \dots a_n)$ to $(\omega f(a_1) f(a_2) \dots f(a_n))$, as seen in the following definition.

DEFINITION

If A and B are Ω-algebras, a map $f : A \to B$ is said to be a ***homomorphism*** *if for all $\omega \in \Omega(n)$ and $a_1, a_2, \dots a_n \in A$,*

$$f(\omega a_1 a_2 \dots a_n) = (\omega f(a_1) f(a_2) \dots f(a_n))$$

When $n = 0$ it is understood that the statement says $f(\omega_A) = (\omega_B)$. A homomorphism that is bijective is called an ***isomorphism****.*

Note that this definition does not regard any equational identities. It never will, as there is no notion of identities being preserved by maps.

EXAMPLES

1. Monoid, group and ring homomorphisms are as usual.

2. If Ω-algebras are pointed sets, a homomorphism $(X, x_0) \to (Y, y_0)$ is a map $f : X \to Y$ satisfying $f(x_0) = y_0$.

3. A homomorphism $f : M \to N$ of left R-modules satisfies $f(a+b) = f(a) + f(b)$ and $f(ka) = kf(a)$ for $a, b \in M, k \in R$. Recall that scalar multiplication is viewed as $|R|$ unary operators. In this case, our definition of a homomorphism certainly matches with this one.

4. If M and N are modules over different rings, they are not the same kind of algebra, so there's no notion of a homomorphism from M to N.

There are a few preliminary things to know about homomorphisms:

THEOREM 1.4 *Let A, B, C be Ω-algebras, $f : A \to B$ and $g : B \to C$ homomorphisms. Then:*

(1) *The composite function $gf : A \to C$ is a homomorphism.*

(2) *The identity map $1_A : A \to A$ is an isomorphism.*

(3) *If f is an isomorphism, then so is its inverse $f^{-1}: B \to A$.*

Proof of Theorem 1.4. (1) The statement follows from

$$gf(\omega a_1 a_2 \dots a_n) = g(\omega f(a_1) f(a_2) \dots f(a_n)) = (\omega gf(a_1) gf(a_2) \dots gf(a_n))$$

for all $\omega \in \Omega(n)$, $a_1, a_2, \dots a_n \in A$.

(2) It is clear that 1_A is bijective, and that $1_A(\omega a_1 a_2 \dots a_n) = (\omega a_1 a_2 \dots a_n) = (\omega 1_A(a_1) 1_A(a_2) \dots 1_A(a_n))$.

(3) Since f is a bijection, then so is f^{-1}, and $ff^{-1} = 1_B$ and $f^{-1}f = 1_A$ hold. We need only show that f^{-1} is a homomorphism:

$$f^{-1}(\omega b_1 b_2 \dots b_n) = f^{-1}(\omega ff^{-1}(b_1) ff^{-1}(b_2) \dots ff^{-1}(b_n))$$

$$= f^{-1}f(\omega f^{-1}(b_1) f^{-1}(b_2) \dots f^{-1}(b_n)) = (\omega f^{-1}(b_1) f^{-1}(b_2) \dots f^{-1}(b_n))$$

whenever $\omega \in \Omega(n)$ and $b_1, b_2, \dots b_n \in B$. ∎

An Ω-algebra A is said to be **isomorphic** to an Ω-algebra B — denoted $A \cong B$ — if there exists an isomorphism $A \to B$. Notice that $A \cong A$ by Theorem 1.4(2), and if $A \cong B$ then $B \cong A$ by Theorem 1.4(3). Now suppose $A \cong B$ and $B \cong C$. Then there exist isomorphisms $f : A \to B$ and $g : B \to C$. The map $gf : A \to C$ is a homomorphism by Theorem 1.4(1) and is bijective because f and g are. Hence, gf is an isomorphism and $A \cong C$. Therefore, isomorphism is an equivalence relation.

Now let's cut the isomorphism and get to some pertinent homomorphisms. Let $\{A_\alpha\}$ be an indexed collection of Ω-algebras and $A = \Pi A_\alpha$. Define $p_\alpha : A \to A_\alpha$ by $p_\alpha(a) = a_\alpha$. Then p_α is seen to be a homomorphism because $(\omega a^1 a^2 \dots a^n)_\alpha = (\omega a^1_\alpha a^2_\alpha \dots a^n_\alpha)$ holds. p_α is called a **projection homomorphism**. Section 4 of Chapter 2 explains more about this.

Making a Homomorphism Surjective

If $f : A \to B$ is a homomorphism of Ω-algebras, its image $f(A)$ may not be all of B. However, it is readily seen to be a subalgebra of B, for if $\omega \in \Omega(n)$ and $b_1, b_2, \dots b_n \in f(A)$, each $b_i = f(a_i)$ for some $a_i \in A$. Furthermore,

$$(\omega b_1 b_2 \dots b_n) = (\omega f(a_1) f(a_2) \dots f(a_n)) = f(\omega a_1 a_2 \dots a_n) \in f(A)$$

Hence, $f(A)$ is a subalgebra of B. Conversely, every subalgebra C of B is the image of some homomorphism into B. Define $\iota : C \to B$ by $\iota(c) = c$ for all $c \in C$. [This map is seen to fix its subjects while enlarging the outside world.] Exercise 1 shows that ι is an injective homomorphism. It is called the **canonical monomorphism** [or **injection homomorphism**] of C into B and is sometimes denoted $C \hookrightarrow B$.

The idea of surjectification is to cut the codomain down to a subalgebra containing the image. Important things to know are that, as shown in (3), injectivity is not affected, and as shown in (4), surjectivity comes from cutting the codomain down entirely to the image.

THEOREM 1.5 (SURJECTIFICATION) *Let $f : A \to B$ be a homomorphism of Ω-algebras and C a subalgebra of B. If $\iota : C \to B$ is the canonical monomorphism, then:*

(1) *There exists a homomorphism $f_1 : A \to C$ such that $f = \iota f_1$ [in other words, that $f_1(a) = f(a)$ for all $a \in A$] if and only if $f(A) \subseteq C$.*

If the equivalent conditions in (1) hold, then

(2) *f_1 is unique;*

(3) *f_1 is injective if and only if f is injective;*

(4) *f_1 is surjective if and only if $f(A) = C$.*

You're probably wondering if there's an analogue of this theorem with injectivity and surjectivity exchanged. The answer is yes, but a new concept in the next section is needed for this.

Proof of Theorem 1.5. (1) If $f(A) \subseteq C$, then $f(a) \in C$ for all $a \in A$, so one can clearly define $f_1 : A \to C$ by $f_1(a) = f(a)$. f_1 is readily seen to be a homomorphism. Conversely, if $f_1 : A \to C$ and $f_1(a) = f(a)$ for all a, then $f(a) \in C$, so that $f(A) \subseteq C$.

(2) Suppose $f_1' : A \to C$ is also a homomorphism satisfying $f = \iota f_1'$. Then $\iota f_1 = \iota f_1'$. Since ι is injective, $f_1 = f_1'$ follows, and f_1 is unique.

(3) Since $f_1(a) = f(a)$ for all $a \in A$, $f_1(a) = f_1(b)$ if and only if $f(a) = f(b)$. So if either one of these statements implies $a = b$, the other does. Furthermore, f_1 is injective if and only if f is.

(4) Note that $f_1(A) = f(A)$, because the maps agree on every element of A. Since f_1 is surjective if and only if its image $f_1(A)$ is equal to its codomain C, this statement follows. ■

An important special case of Theorem 1.5 is when f is injective and $f(A) = C$. In that case, the map f_1 is injective and surjective, so that it is an isomorphism. This gives us

COROLLARY 1.6 *If $f : A \to B$ is an injective homomorphism, then $A \cong f(A)$.*

One can think of an injective homomorphism as an embedding for that reason.

Homomorphisms play an important role in many aspects. There are many ways to think of their structure, one of which is in Exercise 3.

EXERCISES

1. Let B be a subalgebra of A and $\iota : B \to A$ the map defined by $\iota(b) = b$ for all $b \in B$. Show that ι is an injective homomorphism of Ω-algebras.

2. If A is an Ω-algebra and $d : A \to A \times A$ is defined by $d(a) = (a, a)$ for all $a \in A$, show that d is an injective homomorphism. [d is called the **diagonal map** on A.] Conclude that $A \cong \{(a, a) \mid a \in A\}$.

3. An **automorphism** of an Ω-algebra A is an isomorphism from A to A. Show that the set Aut A of all automorphisms of A is a group under the operation of function composition. *Remark*: This holds even if A is itself a group. [*Hint*: Follow the paragraph after the proof of Theorem 1.4.]

4. Let G be a group and X a set. A **group action** of G on X is said to be a map $\cdot : G \times X \to X$ satisfying $ab \cdot x = a \cdot (b \cdot x)$ and $e \cdot x = x$ for all $a, b \in G$, $x \in X$. If A is an Ω-algebra, verify that Aut A acts on A given by $\sigma \cdot a = \sigma(a)$.

5. Let A be an Ω-algebra and $T(\Omega)$ be the one-element algebra $\{\epsilon\}$ given by Exercise 9 of Section 1. Prove that there is exactly one homomorphism $A \to T(\Omega)$.

6. (a) Show that the product $A \times B$ seen in Section 1 is isomorphic to that in Section 2.

   (b) Show that $A^S \cong \prod_{s \in S} A$.

7. (a) If $A \cong B$ and $C \cong D$, prove that $A \times C \cong B \times D$.

   Then prove the following statements for Ω-algebras A, B, C:

   (b) $(A \times B) \times C \cong A \times (B \times C)$

   (c) $A \times B \cong B \times A$

   (d) $T(\Omega) \times A \cong A$

8. (a) If $f : A \to B$ is a homomorphism and A_1 is a subalgebra of A, prove that $f(A_1) = \{f(a) \mid a \in A_1\}$ is a subalgebra of B.

   (b) Now suppose B_1 is a subalgebra of B. Show that $f^{-1}(B_1) = \{a \in A \mid f(a) \in B_1\}$ is a subalgebra of A.

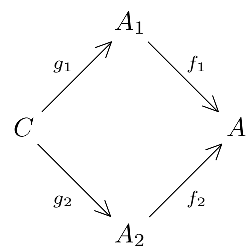
9. (a) If $f : A \to B$ and $g : A \to B$ are homomorphisms, show that $\{a \in A \mid f(a) = g(a)\}$ is a subalgebra of A.

   (b) Let $X \subseteq A$ such that $\langle X \rangle = A$. If $f : A \to B$ and $g : A \to B$ are homomorphisms with $f|X = g|X$, prove that $f = g$. Thus, a homomorphism of A is completely determined by its action on generators of A.

10. Let A_1, A_2 and A be Ω-algebras and $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ homomorphisms. Define $C = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}$.

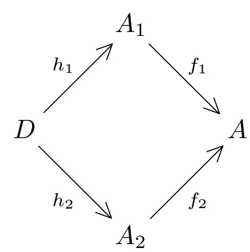
    (a) Show that C is a subalgebra of $A_1 \times A_2$.

(b) Define $g_1 : C \to A_1$ by $g_1(a_1, a_2) = a_1$ and $g_2 : C \to A_2$ by $g_2(a_1, a_2) = a_2$. Show that g_1 and g_2 are homomorphisms, and that $f_1 g_1 = f_2 g_2$; that is, the diagram



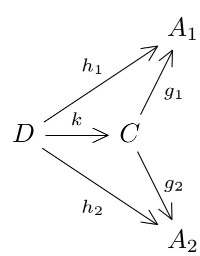
is commutative.

(c) Suppose D is an Ω-algebra and $h_1 : D \to A_1$ and $h_2 : D \to A_2$ are homomorphisms such that



is commutative. Define $k : D \to A_1 \times A_2$ by $k(d) = (h_1(d), h_2(d))$. Show that $k(D) \subseteq C$. Conclude that k can be surjectified to a homomorphism $D \to C$.

(d) Show that k is the unique homomorphism $D \to C$ such that the triangles in



are commutative.

The algebra C and the maps g_1 and g_2 from C are said to be a **pullback** of the maps f_1 and f_2.