1.2 - Subalgebras and Products

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Recall the Ω-algebra from last chapter. What subsets are also Ω-algebras under the operations in Ω? Well, the ones that are closed under the operations. Section 9 shows that any equational identity holding in an Ω-algebra must also hold in each subset closed under the operations. This yields the following definition.

DEFINITION

Let A be an Ω-algebra. A subset B of A is called a ***subalgebra*** *of A provided that whenever $\omega \in \Omega(n)$ and $a_1, a_2, \dots a_n \in B$, then $(\omega a_1 a_2 \dots a_n) \in B$. When $n = 0$, it is understood that B contains every nullary $(\omega_A) \in A$.*

Note: Most authors require every Ω-algebra to be nonempty. This yields differences in the future lessons. I personally think that an Ω-algebra should not be required to be nonempty unless it follows from the equipment. See Exercise 1.

If B is a subalgebra of A, the first pertinent thing to be able to view B as an Ω-algebra itself under the operations in A. Then it is clear that A is a subalgebra of A, and a subalgebra C of a subalgebra B of A is a subalgebra of A.

EXAMPLES

1. Let G be a group and H a subgroup of G. Then H is a group under the operation in G. The converse actually holds in this case: if H is a subset of G which is a group under the binary operation in G, then H is a subgroup of G. This is because, for example, if $a \in G$ and $ab = b$ for at least *one* $b \in G$, then $a = e$.
2. If M is a monoid, a subset of M which is a monoid under M's binary operation need not be a submonoid of M. Consider $M = (\mathbb{Z}_6, \cdot)$, for instance. Let $N = \{[0], [2], [4]\} \subseteq M$. Then N is a monoid under multiplication [with identity [4]], but N is not a submonoid of M because $[1] \notin N$. A submonoid of M must be a monoid under M's multiplication *and* identity element.
3. If Ω-algebras are pointed sets, a subalgebra of (X, x_0) is a subset of X containing x_0.
4. Yeah...let's cut the examples.

Let Sub A be the set of subalgebras of A. We claim that Sub A is a **complete lattice** under inclusion [i.e. a lattice in which every nonempty subset has a sup and an inf]. It all follows from quite a basic lemma:

LEMMA 1.1 *Let L be a partially ordered set with a largest element 1 such that every nonempty subset has an inf. Then L is a complete lattice.*

Proof of Lemma 1.1. We need to show that every nonempty subset $S \subseteq L$ has a sup. Let $\overline{S} = \{a \in L \mid s \leq a \ \forall s \in S\}$ be the set of upper bounds of S. $1 \in \overline{S}$

so $\overline{S}$ is nonempty. Therefore, by hypothesis, $\overline{S}$ has an inf u. Since each $s \in S$ is a lower bound of $\overline{S}$, $s \leq u$ and u is an upper bound of S. If v is any other upper bound of S, then $v \in \overline{S}$, and hence, $u \leq v$. Therefore, u is a sup of S. ■

THEOREM 1.2 *If A is an Ω-algebra, then Sub A is a complete lattice under inclusion.*

Proof of Theorem 1.2. Sub A is clearly a poset under inclusion. Also, $A \in$ Sub A is largest in the poset. Now let $\{A_\alpha\}$ be a nonempty family of subalgebras of A. Exercise 2 shows that the intersection $\cap A_\alpha$ is a subalgebra of A, and it is seen to be the inf of $\{A_\alpha\}$. Therefore, Sub A has a largest element and every nonempty subset has an inf. We can then apply Lemma 1.1 and conclude that Sub A is a complete lattice. ■

According to Theorem 1.2, any family $\{A_\alpha\}$ of subalgebras of A has a least upper bound. This does *not* mean the union $\cup A_\alpha$ is necessarily a subalgebra of A. Rather, it means there is a *subalgebra* of A containing the A_α that's contained in every *subalgebra* of A containing the A_α. This subalgebra is denoted as $\vee A_\alpha$.

However, there *is* a rather important case in which the union $\cup A_\alpha$ is a subalgebra; see Exercise 3.

But unions of subalgebras aren't the only things that can generate subalgebras. In fact, *any* subset of A generates a subalgebra according to the following theorem. Recall that in the proof of Lemma 1.1, we showed that the sup of a set is the inf of its upper bounds. This gives us a clue as to what to do.

For $X \subseteq A$, define the **subalgebra of A generated by X** — denoted $\langle X \rangle$ — to be the intersection of all subalgebras of A containing X. Note that at least one subalgebra of A contains X — namely A itself.

THEOREM 1.3 *Let X be a subset of an Ω-algebra A. Then:*

(1) *$\langle X \rangle$ is a subalgebra of A containing X.*

(2) *Whenever B is a subalgebra of A containing X, $\langle X \rangle \subseteq B$.*

(3) *$\langle X \rangle$ is the only subalgebra of A with properties (1) and (2).*

Proof of Theorem 1.3. (1) $\langle X \rangle$ is the intersection of subalgebras of A, which is a subalgebra of A by Exercise 2. Since X is contained in every operand set, it is contained in the intersection.

(2) If B is a subalgebra of A containing X, then B is one of the operands that $\langle X \rangle$ is the intersection of; hence, $\langle X \rangle \subseteq B$.

(3) Suppose X' is another subalgebra of A satisfying properties (1) and (2). Since X' is a subalgebra of A containing X, then $\langle X \rangle \subseteq X'$ by (2). Reversing the roles, since $\langle X \rangle$ is a subalgebra of A containing X and every such subalgebra contains X', then $X' \subseteq \langle X \rangle$. Therefore, $X' = \langle X \rangle$, and $\langle X \rangle$ is unique. ■

One can think of $\vee A_\alpha$ as $\langle \cup A_\alpha \rangle$ according to the last theorem. It is the smallest subalgebra of A containing every A_α.

Arbitrary Products

We recall the constructions of $A \times B$ and A^S from the last chapter. They are just special cases of the following.

DEFINITION

Let $\{A_\alpha\}$ be an indexed collection of Ω-algebras. Now let A be the set product ΠA_α and for $a \in A$, denote as a_α the component of a in place α. Then A becomes an Ω-algebra where $\omega \in \Omega(n)$ is defined as:

$$(\omega a^1 a^2 \dots a^n)_\alpha = (\omega a^1_\alpha a^2_\alpha \dots a^n_\alpha)$$

for each $a^1, a^2, \dots a^n \in A$ and component index α.

Note that there is no restriction on the cardinality of $\{A_\alpha\}$. The next section shows how $A \times B$ can be viewed as a product of A and B, and that A^S can be viewed as a product of as many A's as elements of S.

When $A_1, A_2, \dots A_n$ are finitely many Ω-algebras, the product is denoted ΠA_i or $A_1 \times A_2 \times \dots \times A_n$.

Note, by the way, that Section 9 shows that identities are safely preserved by the product.

EXERCISES

1. Show that every Ω-algebra is nonempty if and only if $\Omega(0)$ is nonempty [i.e. Ω has a nullary operator].

2. If $\{A_\alpha\}$ is a nonempty family of subalgebras of A, prove that the intersection $\cap A_\alpha$ is a subalgebra of A.

3. Now suppose $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ is an ascending chain of subalgebras of A. Prove that the union $\cup A_i$ is a subalgebra of A. [*Hint*: If $a_1, a_2, \dots a_n \in \cup A_i$, each of them is in one of the algebras in the chain. Why must one of the algebras contain all of them?]

4. Let $A_1, A_2, \dots A_n$ be Ω-algebras. Show that $A = A_1 \times A_2 \times \dots \times A_n$ is finite if and only if all the A_i are, and that $|A| = |A_1||A_2| \dots |A_n|$. Conclude that A is empty if and only if at least one of the A_i is.

5. (a) If $\{A_\alpha\}$ is an indexed collection of Ω-algebras and B_α is a subalgebra of A_α for every α, prove that ΠB_α is a subalgebra of ΠA_α.

   (b) Give an example to show that a subalgebra of ΠA_α need not be described like such.