1.1 - Definition and Examples of Ω-algebras

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(Universal Algebra)

The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.

Operations rise quite often in mathematics. Ever tend to notice how monoids, groups, rings, modules, lattices, etc. have similar abilities? This is because each is a set equipped with certain operations. Any such set can be treated in a generic way, which leads to the following.

Universal algebra studies morphisms, products and a lot of other topics on a generalized operation equipment on a set. To see the idea, we must stick to one batch of operation equipment which we call the *signature*.

Consider the monoid M, for instance. M is a set with an associative binary operation $*$ which has a unit 1. The binary operation $*$ should certainly belong to the signature. Now when we give a set M a binary operation $*$, do we know whether we result in a monoid? Yes, either the operation is associative or it is not, and there is at most one element 1 satisfying $1 * x = x = x * 1$ for every $x \in M$ [see Exercise 1]. So it appears that $*$ is the only operation to belong to the signature. That is not quite true, as we are about to see.

A homomorphism of monoids $f : M \to N$ satisfies $f(1) = 1$ and $f(xy) = f(x)f(y)$ for all $x, y \in M$. If only the binary operation mattered, a homomorphism would only need to satisfy $f(xy) = f(x)f(y)$. This is not sufficient, as there exist maps of monoids that preserve multiplication but do not map 1 to 1. Consider $M = N = (\mathbb{Z}, \cdot)$, for instance. Then if $f : M \to N$ is defined by $f(a) = 0$ for all a, then $f(xy) = f(x)f(y)$ [since $0 \cdot 0 = 0$] but $f(1) \neq 1$.

So both the binary operation $*$ and the unit 1 are needed to keep things in hand. Afterwards, we need only regard the identities $(xy)z = x(yz)$ and $1x = x = x1$, which don't affect homomorphisms at all.

Note that if G and H are *groups* and $f : G \to H$ satisfies $f(xy) = f(x)f(y)$, then f is a group homomorphism [see Exercise 2]. So in terms of group homomorphisms, only the binary operation needs to be regarded. But this is not so for subgroups. The subset $\mathbb{N}$ of the additive group $\mathbb{Z}$ is not a subgroup because $1 \in \mathbb{N}$, but its inverse -1 is not in $\mathbb{N}$. It is closed under addition, nevertheless.

Therefore the group's signature needs to regard the binary operation, the inverse and the identity. It is then straightforward what the requirements of a subgroup would be.

To generalize the idea, it is important to know that a set like that has two things: (1) existential operators; (2) equational identities [axioms of the form $(\dots) = (\dots)$]. Morphisms and subsets that have the equipment need to regard (1), but not (2). (1) comes in the form of *n-ary operators*, which are feed n elements of the set and return an element of the set. As of now, we will stick to only (1).

Let A be a set, and n a nonnegative integer. If ω is a map $A^n \to A$ sending $(a_1, a_2, \dots a_n)$ to $(\omega a_1 a_2 \dots a_n)$, then ω is an n-ary operator on A. An example with $n = 2$ is the binary operation on a monoid. Note that if $n = 0$, ω is just a map from the 1-element set $\{()\}$ to A, which can be thought of as an element

(ω_A) of A. An example of this is the unit 1 of a monoid, which must be regarded by the signature.

There may be many operators in the signature, but each has a certain degree. This motivates the following definitions.

DEFINITION

*A **signature** is a mathematical object Ω such that for each nonnegative integer n, $\Omega(n)$ is a set, whose elements are called n-**ary operators**. Ω can be thought of as $\uplus\Omega(n)$.*

*If Ω is a signature, an Ω-**algebra** is a set A such that for each $n \geq 0$, each $\omega \in \Omega(n)$ is associated with a map $A^n \to A$, where the output under $(a_1, a_2, \ldots a_n)$ is denoted $(\omega a_1 a_2 \ldots a_n)$. The set is called the **carrier** of the Ω-algebra.*

EXAMPLES

A vast majority of the following examples have equational identities. However, it is best that we not generalize the concept of identities until Section 9.

1. Let $\Omega = \{p, 1\}$ where p is binary and 1 is nullary. Then a monoid is an Ω-algebra satisfying the identities $(px(pyz)) = (p(pxy)z)$; $(p(1)x) = x$; $(px(1)) = x$.

2. Let $\Omega = \{p, 1, i\}$ where p is binary, 1 is nullary and i is unary. Then a group is an Ω-algebra satisfying the identities $(px(pyz)) = (p(pxy)z)$; $(p(1)x) = x$; $(px(1)) = x$; $(px(ix)) = (1)$; $(p(ix)x) = (1)$. Note that the last identity is quite redundant; it follows from the other identities.

3. Add the identity $(pab) = (pba)$ to the previous example to get an abelian group. They form a signature of their own.

4. A ring is an Ω-algebra with even more identities, where $\Omega(2) = \{s, p\}$ (s sum, p product), $\Omega(1) = \{n\}$ (additive inverse) and $\Omega(0) = \{0, 1\}$. As an exercise, write out all the necessary identities; one of them is $(px(syz)) = (s(pxy)(pxz))$.

5. A **rng** is an Ω-algebra where $\Omega(2) = \{s, p\}$, $\Omega(1) = \{n\}$ and $\Omega(0) = \{0\}$, and all the ring's identities that don't involve 1 are satisfied. A rng can be thought of as a "ring without unit."

6. A **ring with involution** is a ring R with an extra unary operator $a \to a^*$ satisfying $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, $1^* = 1$ and $(a^*)^* = a$. It is easily seen that there is a signature for rings with involution.

7. If $\Omega(n) = \emptyset$ for all n, then an Ω-algebra is simply a set. You can think of this as a set equipped with no operations at all. Ω is called the **empty signature**.

8. A **pointed set** is an Ω-algebra where Ω consists of a single nullary operator for the **base point**. It can be thought of as a pair (X, x_0), where $x_0 \in X$.

9. A **set with involution** is an Ω-algebra A where Ω consists of a single unary operator * and $(a^*)^* = a$ for all $a \in A$. The operator is called an **involution**.

10. Let R be a fixed ring. Define $\Omega(2) = \{s\}$, $\Omega(0) = \{0\}$ and $\Omega(1) = R$. Then an Ω-algebra satisfying the correct identities [such as $(r(sx)) = ((rs)x)$ when $r, s \in R$] is a left R-module. Note that there is no restriction on the cardinality of operators or identities.

Warning: There is no signature for all modules. The modules *over a given ring* can be put into a signature. It is pertinent to know that there's no such thing as a module homomorphism from M to N if they are modules over entirely different rings.

11. In a similar way right R-modules and R-S-bimodules can be defined.

12. A **Lie algebra** over a commutative ring R is an R-module L with a binary operator $a, b \to [a : b]$ satisfying $[x : (y + z)] = [x : y] + [x : z]$, $[(x + y) : z] = [x : z] + [y : z]$, $[x : cy] = c[x : y] = [cx : y]$, $[x : x] = 0$ and $[x : [y : z]] + [y : [z : x]] + [z : [x : y]] = 0$. Once again, this is an Ω-algebra with identities, and they will be dealt with in Sections 9 and up.

13. An **associative algebra** over a commutative ring R is a ring A which is an R-module with the same addition, such that $(cx)y = c(xy) = x(cy)$ for all $c \in R$, $x, y \in A$. For example, the matrix ring $M_n(R)$ is an associative algebra over R.

14. A **magma** is a set equipped with a binary operation. It does not require any identities. A **semigroup** is a magma whose operation is associative; i.e. satisfies the identity $(pa(pbc)) = (p(pab)c)$. Thus a monoid is a semigroup with an identity element.

15. A **lattice** is an Ω-algebra where $\Omega(2) = \{\wedge, \vee\}$ satisfying $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$, $a \wedge b = b \wedge a$, $a \vee b = b \vee a$, $a \wedge a = a = a \vee a$, $a \wedge (a \vee b) = a = a \vee (a \wedge b)$.

16. Let M be a monoid. An M-action is an Ω-algebra X with $\Omega(1) = M$ [that is, a set X along with a map $M \times X \to X$] satisfying $1x = x$ and $(mn)x = m(nx)$ for $m, n \in M$, $x \in X$. Thus a set with involution is an M-action where M is the group $\mathbb{Z}_2$.

You should be convinced that there are loads of different kinds of Ω-algebras. This is why we should be able to give general proofs that work for all of them.

New Signatures from Old Ones

If Ω_1 and Ω_2 are signatures, Ω_2 is said to be an **extension** of Ω_1 provided that $\Omega_1(n) \subseteq \Omega_2(n)$ for all n. In this case, every Ω_2-algebra is an Ω_1-algebra.

EXAMPLES

Note that the definition of an extension can be rephrased when identities are involved. But like we said, we are not dealing with identities quite yet.

1. The group's signature is an extension of the monoid's, which is an extension of the semigroup's, which is an extension of the magma's.

2. The ring's signature is an extension of the rng's, where the identity is added. The signature for the ring with involution is an extension of the ring's

signature. The rng's signature is an extension of the signature for an abelian group (because a rng is an abelian group under addition).

3. Every signature is an extension of the empty signature for sets, because $\emptyset \subseteq \Omega(n)$ whenever Ω is a signature. This works as promised; an Ω-algebra is a set.

4. If F is a field, then associative algebras over F are an extension of rings, and also an extension of vector spaces over F. Lie algebras over F are an extension of vector spaces over F.

5. Note that a monoid is actually an extension of a pointed set, because a monoid M can have the weaker treatment as a pointed set with base point 1.

6. Abelian groups are an extension of groups, and commutative rings are an extension of rings.

New Ω-algebras from Old Ones

It is high time we stop talking about all the different signatures, and from this point, focus on a single signature Ω. Can two Ω-algebras A and B be combined, in a generic way that doesn't depend on Ω? The answer is yes: we define $A \times B$ to be the usual Cartesian product of sets, and for each $n \in \Omega(n)$, we define

$$(\omega(a_1, b_1)(a_2, b_2) \dots (a_n, b_n)) = ((\omega a_1 a_2 \dots a_n), (\omega b_1 b_2 \dots b_n))$$

For instance, if A and B are sets with involution, $A \times B$ is defined by $(a, b)^* = (a^*, b^*)$.

Now suppose A is an Ω-algebra and S is a set. [S need not be an Ω-algebra at all.] One can define an Ω-algebra structure on the set A^S of functions $S \to A$ thus: for $\omega \in \Omega(n)$ and $f_1, f_2, \dots f_n \in A^S$, $(\omega f_1 f_2 \dots f_n) : S \to A$ is defined by $(\omega f_1 f_2 \dots f_n)(s) = (\omega f_1(s) f_2(s) \dots f_n(s))$.

This is a vague introduction to Ω-algebras. It should be easy to remember in the future sections.

EXERCISES

1. If M is a set equipped with a binary operation $*$, prove that there is at most one $1 \in M$ such that $1 * x = x = x * 1$ for every $x \in M$.

2. Let G and H be groups. If $f : G \to H$ such that $f(xy) = f(x)f(y)$ for all $x, y \in G$, prove that f is a homomorphism. [You need to show that $f(e) = e$ and $f(x^{-1}) = f(x)^{-1}$ for all $x \in G$.]

3. What is wrong with the following argument that subgroups of a group need not regard the identity element? "If H is a subgroup of G and $a \in H$, then $a^{-1} \in H$ since the inverse of an element of H is also in H. Furthermore, $aa^{-1} = e \in H$ since H is closed under multiplication. Therefore, every subset of G closed under multiplication and inverses automatically contains the identity element."

4. Let G be a semigroup.

   (a) If there exists $e \in G$ such that $ea = a$ for all a and for each $a \in G$, there exists $d \in G$ such that $da = e$, prove that G is a group.

   (b) If there exists $e \in G$ such that $ea = a$ for all a and for each $a \in G$, there exists $d \in G$ such that $ad = e$, show by example that G may not be a group.

   (c) If G is finite and nonempty, and whenever $ab = ac$ or $ba = ca$ then $b = c$, prove that G is group.

   (d) Show by example that (c) may be false if G is infinite.

   (e) If G is nonempty and for all $a, b \in G$, there exist $x, y \in G$ such that $ax = b = ya$, prove that G is a group.

5. For a Boolean algebra, what are all the operations needed in the signature?

6. Show that an associative algebra A over a commutative ring R is a Lie algebra given by $[a : b] = ab - ba$ for $a, b \in A$.

7. If L is a Lie algebra over R, prove that $[a : b] = -[b : a]$ for $a, b \in L$. [*Hint*: Expand $[(a + b) : (a + b)]$.]

8. (a) Show that every signature Ω is an extension of Ω.

   (b) Show that if Ω_3 is an extension of Ω_2 and Ω_2 is an extension of Ω_1, then Ω_3 is an extension of Ω_1.

9. Let $T(\Omega)$ be the 1-element set $\{\epsilon\}$, where for each $\omega \in \Omega(n)$, $(\omega\epsilon\epsilon \ldots \epsilon) = \epsilon$. Convince yourself that $T(\Omega)$ is an Ω-algebra. It is called the **terminal** or **trivial Ω-algebra**.